

STEADY STATE RESPONSE OF A NONLINEAR SYSTEM

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ABSTRACT. This paper presents a method of the determination of the steady state response for a class of nonlinear systems. The response of a nonlinear system to a given input is first obtained in the form of a series solution in the multi-dimensional frequency domain. Conditions are then determined for which this series solution will converge. The conversion from multidimensions to a single dimension is then made by the method of association of variables, and thus an equivalent linear model of the nonlinear system is obtained. The steady state response is then found by any technique employed with linear system.

KEYWORDS AND PHRASES. *Association of variables, Multidimensional frequency domain, Nonlinear transfer function.*

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1. INTRODUCTION.

This paper considers a nonlinear system described by the equation of the form

$$Ly + a_3y^3 + a_5y^5 = x(t) \quad (1.1)$$

L is a linear operator defined as

$$L(s) = b_n s^n + b_{n-1} s^{n-1} + \dots + b_0$$

$a_3, a_5; b_n$ are constants. $x(t)$ is the input to the nonlinear system and $y(t)$ is the response to it. Many physical systems encountered in practice can be related to above kind of equation. We shall investigate the conditions under which a bounded input to the system will produce a bounded output. This will lead to the stability conditions of the above type of nonlinear system.

2. ANALYSIS OF A NONLINEAR SYSTEM

Equation (1.1) can be viewed as follows: Volterra [1] has shown that the response $y(t)$ can be represented as some functional of the input $x(t)$. The following functional expansion was suggested:

$$y(t) = \int h_1(t-\tau_1)x(\tau_1)d\tau_1 + \iint h_2(t-\tau_1, t-\tau_2)x(\tau_1)x(\tau_2)d\tau_1d\tau_2 + \quad (2.1)$$

We assume a causal system, i.e., $x(t) = 0$ for $t < 0$. The limits of integrations are $[0, \infty]$. Let

$$y_n(t) = \int \dots \int_{n\text{-fold}} h_n(t-\tau_1 \dots t-\tau_n)x(\tau_1)x(\tau_2) \dots x(\tau_n)d\tau_1d\tau_2 \dots d\tau_n$$

Then equation (2.1) can be written as

$$y(t) = \sum_{n=1}^{\infty} y_n(t) \quad (2.2)$$

$h_1(t-\tau_1)$ is the 1st order Volterra Kernel and $h_n(t-\tau_1 \dots t-\tau_n)$ is the nth order Volterra Kernel.

3. DETERMINATION OF THE VOLTERRA KERNELS

By applying the technique or reversion of algebraic series [2] from (1.1) we get

$$y = L^{-1}x - a_3 L^{-1}(L^{-1}x)^3 + L^{-1}\{3a_3^2 L^{-1}(L^{-1}x)^5 - a_5 (L^{-1}x)^5\} + \tag{3.1}$$

It is clearly understood that x and y are both functions of time t. Let us define

$$L^{-1}x \equiv \int h(t-\tau)x(\tau)d\tau \tag{3.2}$$

where h(t) is the impulse response of the linear part of the system. Hence from (2.2)

$$\begin{aligned} y_1 &= \int h(t-\tau)x(\tau)d\tau \\ y_3 &= -a_3 L^{-1}\left(\int h(t-\tau)x(\tau)d\tau\right)^3 \\ &= -a_3 L^{-1} \iiint h(t-\tau_1)h(t-\tau_2)h(t-\tau_3)x(\tau_1)x(\tau_2)x(\tau_3)d\tau_1 d\tau_2 d\tau_3 \\ &= -a_3 \iiint h(t-u)h(u-\tau_1)h(u-\tau_2)h(u-\tau_3)x(\tau_1)x(\tau_2)x(\tau_3)d\tau_1 d\tau_2 d\tau_3 du \end{aligned}$$

Similarly

$$\begin{aligned} y_5 &= 3a_3^2 \int \dots \int h(t-v)h(v-u)h(u-\tau_1)\dots h(u-\tau_5)x(\tau_1)\dots x(\tau_5)d\tau_1 \dots d\tau_5 dudv \\ &\quad - a_5 \int \dots \int h(t-u)h(u-\tau_1)\dots h(u-\tau_5)x(\tau_1)\dots x(\tau_5)d\tau_1 \dots d\tau_5 du. \end{aligned}$$

Hence by comparing (2.1) and (2.2)

$$h_1(t-\tau) = h(t-\tau)$$

$$h_2(t-\tau_1, t-\tau_2) = 0$$

$$h_3(t-\tau_1, t-\tau_2, t-\tau_3) = -a_3 \int h(t-u)h(u-\tau_1)h(u-\tau_2)h(u-\tau_3)du$$

$$h_4(t-\tau_1 \dots t-\tau_4) = 0$$

$$h_5(t-\tau_1 \dots t-\tau_5) = 3a_3^2 \iint h(t-u)h(v-u)h(u-\tau_1) \dots h(u-\tau_5)dudv \\ - a^5 \int h(t-u)h(u-\tau_1) \dots h(u-\tau_5)du.$$

Higher order kernels can thus be determined by expanding further the equation (3.1).

4. CONVERGENCY OF THE FUNCTIONAL EXPANSION

Let

$$\max_{0 < t \leq \infty} |x(t)| \leq X$$

$$y_1 \leq \int |h(t-\tau)| |x(\tau)| d\tau \leq \int |h(t)| dt \cdot X = HX$$

where

$$\int |h(t)| dt = H < \infty$$

$$y_3 \leq |a_3| \int |h(t-\tau_1)| |x(\tau_1)| d\tau_1 \int |h(t-\tau_2)| |x(\tau_2)| d\tau_2 \int |h(t-\tau_3)| |x(\tau_3)| d\tau_3 \int |h(t-u)| du.$$

or

$$y_3 \leq |a_3| \left(\int |h(t)| dt \right)^4 X^3 = |a_3| H^4 X^3$$

Similarly

$$y_5 \leq (3a_3^2H + |a_5|)H^6X^5$$

Let

$$Y = HX + |a_3|H^4X^3 + (3a_3^2H + |a_5|)H^6X^5 \tag{4.1}$$

Hence

$$|y_i| < |Y_i|$$

Again by applying the technique of reversion of series [3] it is easy to show that (4.1) is the solution of (4.2)

$$Y = HX + H(|a_3|Y^3 + |a_5|Y^5) \tag{4.2}$$

and

$$X = \frac{1}{H} Y - |a_3|Y^3 - |a_5|Y^5 \tag{4.3}$$

Let $X = X_1$ and $Y = Y_1$ corresponding to the value $\frac{dY}{dX} = \infty$. Hence if $0 < X \leq X_1$ then $0 < Y \leq Y_1$. Therefore in this above region a bounded input will produce a bounded output. Y_1 can be determined from

$$5H|a_5|Y_1^4 + 3H|a_3|Y_1^2 - 1 = 0 \tag{4.4}$$

corresponding to

$$\frac{dY}{dX} = \infty$$

or

$$Y_1^2 = \frac{-3H|a_3| \pm \sqrt{9H^2a_3^2 + 20H|a_5|}}{10H|a_5|} \tag{4.5}$$

considering only the positive value of Y_1^2 , Y_1 is found from $Y_1 = \sqrt{Y_1^2}$.

From (4.3)

$$X_1 = \frac{1}{H} Y_1 - |a_3| Y_1^3 - |a_5| Y_1^5 \quad (4.6)$$

Therefore for $x(t) < X_1$, the functional expansion (2.1) will be convergent.

REMARKS:

With the assumed solution of equation (1.1)

$$y(t) = \sum_{n=1}^{\infty} \int \dots \int h_n(\tau_1, \tau_2, \dots, \tau_n) x(t-\tau_1) \dots x(t-\tau_n) d\tau_1 \dots d\tau_n$$

$$Y = \sum_{n=1}^{\infty} C_n X^n$$

where

$$C_n = \int \dots \int |h(\tau_1, \tau_2, \dots, \tau_n)| d\tau_1 \dots d\tau_n$$

And for the time invariant system, C_n is independent of time [3] which has the radius of convergence X_1 , which in turn implies the convergence of the Volterra series (2.1) for

$$\max |x(t)| < X_1$$

$$0 < t \leq \infty$$

5. REVIEW OF MULTIDIMENSIONAL TRANSFORM.

As an analogy to the linear system, the nth order transform of a nonlinear system [4] is defined to be

$$H_n(s_1, s_2, \dots, s_n) = \int \dots \int h_n(\tau_1, \tau_2, \dots, \tau_n) \exp[-s_1 \tau_1 + s_2 \tau_2 + \dots + s_n \tau_n] d\tau \dots d\tau_n \quad (5.1)$$

One method of determining $H_n(s_1, s_2, \dots, s_n)$ is the harmonic input method [5]. When the system described by (1.1) is excited with a set of n unit amplitude exponentials at the noncommensurate frequencies s_1, s_2, \dots, s_n

$$x(t) = \exp(s_1 t) + \exp(s_2 t) + \dots + \exp(s_n t) \quad (5.2)$$

the output will contain exponential components of the form

$$y(t) = \sum_{n=1}^{\infty} \sum_M \frac{H_n(s_1, s_2, \dots, s_n)}{m_1! m_2! \dots m_n!} \exp(m_1 s_1 t + m_2 s_2 t + \dots + m_n s_n t) \quad (5.3)$$

while M under the summation indicates that for each n the sum is to be taken over all distinct values of m_n where $1 \leq m_1 \leq m_2 \leq \dots \leq m_n$ such that $m_1 + m_2 + \dots + m_n = n$.

Thus to determine $H_n(s_1, s_2, \dots, s_n)$ we substitute the values of $x(t)$ and $y(t)$ from (5.2) and (5.3), respectively, in the system equation (1.1) and equate the coefficients of

$$n! \exp(s_1 + s_2 + \dots + s_n) t$$

By substituting the values of $y(t)$ and $x(t)$ from (5.3) and (5.2), respectively, in (1.1) the first three nonzero transfer functions are found to be:

$$H_1(s_1) = \frac{1}{L(s_1)} \quad (5.4)$$

$$H_3(s_1, s_2, s_3) = \frac{-a_3}{L(s_1 + s_2 + s_3) L(s_1) L(s_2) L(s_3)} \quad (5.5)$$

$$H_5(s_1, \dots, s_5) = - \frac{a_3}{L(s_1 + s_2 + \dots + s_5) L(s_1 + s_2 + s_3) L(s_4) L(s_5)} + \frac{a_5}{L(s_1 + s_2 + \dots + s_5) L(s_1) L(s_2) \dots L(s_5)} \quad (5.6)$$

Let A mean associate variables s_1, s_2, \dots, s_n . Then

$$Y(s) = H_1(s)X(s) + A\{H_3(s_1, s_2, s_3)X(s_1)X(s_2)X(s_3)\} + A\{H_5(s_1, \dots, s_5)X(s_1)\dots X(s_5)\} + \quad (5.7)$$

Now by applying the Final Value Theorem we get

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s)$$

Subject to the convergency of (2.1) it is often enough to consider the first two nonzero terms of (5.7) to represent the system in the frequency domain.

6. EXAMPLE.

We consider a nonlinear system of the form (1.1) whose linear transfer function is given by

$$\frac{1}{L(s)} = H(s) = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Let its impulse response be $h(t)$.

It is easy to see for $\zeta \leq 0$, $\int |h(t)| dt = H$ is unbounded. Hence the Volterra series expansion (2.1) will be divergent, and the system will not have any bounded steady state response. If $\zeta \geq 1$ it is easy to evaluate $\int |h(t)| dt$ since $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ will have real roots. If H is bounded, convergency of the Volterra series can further be investigated. If $0 < \zeta < 1$ the evaluation of H is a little more involved.

For $0 < \zeta < 1$

$$h(t) = \frac{e^{-\zeta\omega_n t}}{\omega_n \sqrt{1-\zeta^2}} \sin(\omega_n \sqrt{1-\zeta^2} t) \quad [7]$$

Let

$$\alpha = \zeta\omega_n, \quad \beta = \omega_n \sqrt{1-\zeta^2}$$

hence

$$h(t) = \frac{e^{-\alpha t}}{\beta} \sin \beta t$$

We consider the integral

$$H' = \int e^{-\alpha t} |\sin \beta t| dt$$

Now

$$\int e^{ax} \sin px \, dx = \frac{e^{ax}(a \sin px - p \cos px)}{a^2 + p^2}$$

$\sin \beta t < 0$ in the intervals $\left\{\frac{\pi}{\beta}, \frac{2\pi}{\beta}\right\}$, $\left\{\frac{3\pi}{\beta}, \frac{4\pi}{\beta}\right\}$, $\left\{\frac{5\pi}{\beta}, \frac{6\pi}{\beta}\right\}$, etc. Hence

$$H' = \frac{\beta}{\alpha^2 + \beta^2} \left[\left\{e^0 + e^{-m}\right\} + \left\{e^{-m} + e^{-2m}\right\} + \left\{e^{-2m} + e^{-3m}\right\} + \dots \right]$$

$$= \frac{\beta}{\alpha^2 + \beta^2} \left[2e^0 + 2e^{-m} + 2e^{-2m} + \dots - e^0 \right]; \quad m = \frac{\alpha\pi}{\beta}$$

$$H' = \frac{\beta}{\alpha^2 + \beta^2} \left[\frac{2}{1 - e^{-n}} - 1 \right] = \frac{\beta}{\alpha^2 + \beta^2} \cdot \frac{1 + e^{-m}}{1 - e^{-m}}$$

Now $-\coth(-u) = \coth u = \frac{1 + e^{-2u}}{1 - e^{-2u}}$, hence

$$H' = \frac{\beta}{\alpha^2 + \beta^2} \coth\left(\frac{m}{2}\right)$$

Therefore,

$$H = \frac{1}{\omega_n^2} \coth\left(\frac{\pi\zeta}{2\sqrt{1-\zeta^2}}\right)$$

As a numerical example we consider the following system:

$$\ddot{y} + \dot{y} + 3y + \tanh y = A \cos \omega t \quad (6.1)$$

By expanding $\tanh y$ and by retaining the first three terms we get

$$\ddot{y} + \dot{y} + 4y - \frac{1}{3}y^3 + \frac{2}{15}y^5 = A \cos \omega t \quad (6.2)$$

The above approximation is quite valid if $y < 1$. Here

$$\zeta = \frac{1}{4}, \quad \omega_n = 2$$

Hence

$$H = \frac{1}{4} \coth \left(\frac{\pi \times 0.25}{2 \times 0.96825} \right) = 0.64984$$

$$Y_1^2 = \frac{-3 \times 0.64984 \times \frac{1}{3} \pm \sqrt{9 \times (0.64984)^2 \cdot \frac{1}{9} + 20 \times 0.64984 \times \frac{2}{15}}}{10 \times 0.64984}$$

$$= +0.94432 \text{ hence } Y_1 = 0.9717658$$

$$X_1 = \frac{0.9717658}{0.64984} - \frac{1}{3} (0.9717658)^3 - \frac{2}{15} (0.9717658)^5$$

$$= 1.49539 - 0.305888 - 0.1155438 = 1.07395$$

Hence the system will have a steady response for $|A \cos \omega t| < 1.07395$. Response of the linear part = $\lim_{s \rightarrow 0} \frac{1}{s^2 + s + 4} \cdot \frac{As^2}{s^2 + \omega^2} = 0$. It is not necessary to determine the response of the nonlinear parts because being convergent, the other terms of the Volterra expansion can not exceed in absolute value of the linear term. Hence the system will be stable for $|x(t)| < |1.07395 \cos \omega t|$. If $x(t) = A$, then for $A < 1.07309$, the approximate value of the steady state response is given by

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} y_1(t) + \lim_{t \rightarrow \infty} y_3(t)$$

$$\lim_{t \rightarrow \infty} y_1(t) = \lim_{s_1 \rightarrow 0} \frac{s_1}{s_1^2 + 2\zeta\omega_n s_1 + \omega_n^2} \cdot \frac{A}{s_1} = \frac{A}{\omega_n^2}$$

In order to determine $\lim_{t \rightarrow \infty} y_3(t)$ we first obtain $Y_3(s_1, s_2, s_3)$

$$Y_3(s_1, s_2, s_3) = H_3(s_1, s_2, s_3)X(s_1)X(s_2)X(s_3)$$

From (5.5)

$$Y_3(s_1, s_2, s_3) = - \frac{a_3}{L(s_1+s_2+s_3)L(s_1)L(s_2)L(s_3)} \frac{A^3}{s_1 s_2 s_3}$$

Where

$$L(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Now by the technique of association of variables [7,8,9] $Y_3(s_1, s_2, s_3)$ is translated in one frequency domain, thus we obtain

$$Y_3(s) = - \frac{A^3 a_3}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \left[\frac{1}{\omega_n^4 s (s^2 + 2\zeta\omega_n s + \omega_n^2)} + \frac{1}{A_1} + \dots + \frac{1}{A_8} \right]$$

$\frac{1}{A_1}, \frac{2}{A_2} \dots \frac{1}{A_8}$ are functions of (s, ζ, ω_n) but do not contain any factor of the form $\frac{1}{s}$. Hence

$$\begin{aligned} \lim_{t \rightarrow \infty} y_3(t) &= - \lim_{s \rightarrow 0} \frac{s A^3 a_3}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \cdot \frac{1}{\omega_n^4 s (s^2 + 2\zeta\omega_n s + \omega_n^2)} \\ &\quad + \lim_{s \rightarrow 0} s \left(\frac{1}{A_1} + \dots + \frac{1}{A_8} \right) \end{aligned}$$

$$\lim_{t \rightarrow \infty} y_3(t) = -\frac{a_3 A^3}{\omega_n^2} + 0$$

The steady state response of the system is approximately given by

$$y_{ss} = \frac{A}{\omega_n^2} \left(1 - \frac{A a_3}{\omega_n^2} \right)$$

For

$$a_3 = -\frac{1}{3}, \quad \omega_n^2 = 4, \quad A = 1$$

$$y_{ss} = \frac{1}{4} \left(1 + \frac{1}{3 \cdot 64} \right) = \frac{1}{4} + \frac{1}{768}$$

If $x(t) = \delta(t)$. Again by associating the variables [8] we get

$$y_{ss} = \lim_{s \rightarrow 0} \frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \lim_{s \rightarrow 0} \left(\frac{1}{B_1} + \frac{1}{B_2} + \frac{1}{B_3} \right) = 0$$

Since $\frac{1}{B_1}$, $\frac{1}{B_2}$, $\frac{1}{B_3}$ do not contain any factor of the form $\frac{1}{s}$.

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