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SPIRAL LIKE INTEGRAL OPERATORS

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ABSTRACT. In this paper we investigate the Robertson-Libera integral operators for the class of spiral like univalent and analytic functions. We find that special types of transformations preserve the class property. Our results generalize or sharpen the results recently obtained by Miller et al [7] and by Causey and White [3].

KEY WORDS AND PHRASES: *Univalent analytic functions, Starlike; convex.*
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1. INTRODUCTION.

Let $R^+ = \{x \mid x \geq 0\}$, $D = \{z \mid |z| < 1\}$, C = the complex numbers, $S = \{f|f:D \to C,\ f$ to be regular, univalent and $f(0) = o,\ f'(0) = 1\}$, $K_\beta = \{f|f \in S$ and $\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta;\ 0 \leq \beta < 1\}$, $S(\alpha,\beta) = \{f|f \in S,$

$\mathrm{Re}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > \beta\cos\alpha,\ 0 \le \beta < 1,\ \frac{-\pi}{2} \le \alpha \le \frac{\pi}{2}\},\ S_\beta^* = S(0,\beta),\ S^* = S(0,0)$, $S(\alpha,0) = S(\alpha)$ and finally $I^+ = \{n|n = 1, 2, 3, \ldots\}$. f is said to be starlike of order β with respect to the origin if and only if $f \in S_\beta^*$ and f is α-spiral like of order β if and only if $f \in S(\alpha,\beta)$.

In 1964, Robertson [10], proved the following

THEOREM A: - Let k(z) denote the Koebe function $z(1-z)^{-2}$, which is univalent and starlike with respect to the origin for $|z| < 1$. Then the function

$$S(z) = \frac{2}{z}\int_0^z k(t)dt \tag{1.1}$$

is also univalent and starlike with respect to the origin for $|z| < 1$. The path of integration is restricted in the obvious way.

THEOREM B: - Let $S(z)$ and k(z) be defined as in theorem A. Then

$$T(z) = [k(z) - S(z)]^{1/2} \tag{1.2}$$

is univalent and starlike in D and satisfies the inequality

$$|T(z)| \le \left(\frac{1-|z|}{|1-z|}\right)T(|z|) = [zk(z)]^{1/2}. \tag{1.3}$$

Libera [6] established that theorem A holds true when k(z) is replaced by any $f \in S^*$. Bernardi [2] greatly generalised Libera's result. Many authors studied the operators of the form

$$F_f(z) = \frac{(1+\gamma)}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt, \tag{1.4}$$

where γ is a real constant and f belongs to some favoured class of functions from S (see, ref. [1]-[14]). Recently, operators (1.4) have been studied in more general form by Causey and White [3] and Miller, Mocanu and Reade [7], independently.

THEOREM C [3]. Suppose $f \in S^*$, $g \in \bar{K}$, $\alpha, c \in I^+$, $\gamma, \delta \in R^+$ and $(2\delta+\gamma) \le \min(2\alpha, 2c)$. Then the function F defined by

$$F(z) = \left[cz^{\alpha-c} \int_0^z t^{c-1}\left(\frac{f(t)}{t}\right)^{\delta}\left(\frac{g(t)}{t}\right)^{\gamma} dt\right]^{1/\alpha}, \tag{1.5}$$

belongs to S^*.

THEOREM D ([7], THM 6, P. 165): - Let $\xi, \beta^*, \gamma, \rho$ and δ be real constants satisfying the conditions $\xi \ge 0$, $\beta^* > 0$, $\xi + \delta = \beta^* + \gamma > 0$ and

$$0 \le \frac{\rho}{2} \le \begin{cases} \delta & \text{if} \quad \gamma \le 0 \\ \min\{\delta, \delta-\gamma+\frac{1}{2}\min(\frac{\beta^*}{\gamma}, \frac{\gamma}{\beta^*})\} & \text{if} \quad \gamma > 0. \end{cases} \tag{1.6}$$

If $f \in S^*$, $g \in \bar{K}$ then the function

$$F(z) = \left[\frac{\beta^*+\gamma}{z^{\gamma}} \int_0^z f^{\xi}(t) g^{\rho}(t) t^{\delta-\rho-1} dt\right]^{1/\beta^*} \tag{1.7}$$

belongs to S^*.

Theorem C is more or less contained in Theorem D. The proof of Theorem C, due to Causey and White depends on a result of Sakaguchi ([11], p. 74), whereas the proof of Theorem D, depends on a lemma due to Lewandowski, Miller and Zlotkiewicz ([5], p. 114). Since this latter lemma is more powerful than the result of Sakaguchi ([11], p. 74), Theorem D is more general than that of Theorem C. In this paper we shall prove certain theorems which in a way are sharper than Theorems C and D, in the process generalizing the main theorem of Miller, Mocanu and Reade [7]:

THEOREM E: - Let $\psi(z) = 1 + \ldots, \phi(z) = 1+\ldots,$ be functions defined in D with the property $\phi(z)\psi(z) \neq 0$ there. Let α, β, γ, δ be real constants such that $\alpha \geq 0$, $\beta > 0$, $\delta \geq 0$, $\alpha+\delta > 0$ and $\alpha+\delta = \beta+\gamma$. If there exists a non-negative constant J that satisfies

$$J \geq \gamma+\mathrm{Re}\left\{\frac{z\psi'(z)}{\psi(z)}\right\}, \ \beta+\gamma > J \text{ and } \delta+\mathrm{Re}\left\{\frac{z\phi'(z)}{\phi(z)}\right\} \geq \max[0, \ J-\lambda(J)]$$

where $\lambda(J) = \frac{1}{2}\max[(\beta+\gamma-J)/J, \ J/(\beta+\gamma-J)]$, $\lambda(0) = 0$ and if $f \in S^*$, then there exists a function

$$F(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\psi(z)}\int_0^z f^{\alpha}(t)\phi(t)t^{\delta-1}dt\right]^{1/\beta} \in S^*. \tag{1.8}$$

Our generalizations of these theorems depend on the following result of Jack [4] which is also due to Suffridge [14].

LEMMA A: - Suppose that w is analytic for $|z| \leq r < 1$, $w(0) = 0$ and $|w(z_1)| = \max_{|z|=r} |w(z)|$. Then

$$z_1 w'(z_1) = K w(z_1) \quad \text{and} \quad K \geq 1. \tag{1.9}$$

Ways of generalizing Theorem A have been obtained in ([3], p. 458) and ([7], p.157). We shall investigate generalizing Theorem B similarly. Perhaps these generalizations of Theorem B have not been investigated because application of Sakaguchi's results is not easy. However the lemma of Jack is easy and powerful enough to apply in various situations.

2. FUNDAMENTAL THEOREMS

Our generalization of theorems A, C, and D is the following.

THEOREM 1: - Let $f_n \in S(\alpha,\beta_n)$, $n = 0,1,2,\ldots$, $(f_o = f,\ \beta_o = \beta)$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, $0 \le \beta_n < 1$, a, b, $\lambda_n \in R^+$ and $\{-b\beta + \sum_{n=1}^{M} \lambda_n(1-\beta_n)\}\cos\alpha \le \min\{\mathrm{Re}\ c,\ a\cos\alpha\}$. Then the function

$$F(z) = \left[\left(\frac{c+be^{i\alpha}}{z^{c-ae^{i\alpha}}}\right)\int_0^z t^{c-1}\left[f(t)\right]^{be^{i\alpha}}\left\{\prod_{n=1}^{M}\left(\frac{f_n(t)}{t}\right)^{\lambda_n e^{i\alpha}}\right\}dt\right]^{\left(\frac{e^{-i\alpha}}{a+b}\right)} \tag{2.1}$$

belongs to $S(\alpha, \frac{a+b\beta-\sum_{n=1}^{M}\lambda_n(1-\beta_n)}{a+b})$. In (2.1) all powers are principal ones.

PROOF OF THEOREM 1: - Let us write

$$G(t) = t^{c-1}\left[f(t)\right]^{be^{i\alpha}}\left\{\prod_{n=1}^{M}\left(\frac{f_n(t)}{t}\right)^{\lambda_n e^{i\alpha}}\right\} \tag{2.2}$$

$$e^{i\alpha}\frac{zF'(z)}{F(z)} = \left(\frac{m\cos\alpha}{a+b}\right) + i\sin\alpha + \left(\frac{1+w(z)}{1-w(z)}\right)\left(\frac{a+b-m}{a+b}\right)\cos\alpha \tag{2.3}$$

where

$$m = b\beta + a - \sum_{n=1}^{M}\lambda_n(1-\beta_n). \tag{2.4}$$

Then by (2.1) via differentiation, we have

$$z^{c-ae^{i\alpha}-1}\left[F(z)\right]^{(a+b)e^{i\alpha}}\left\{(c-ae^{i\alpha})+(a+b)e^{i\alpha}\frac{zF'(z)}{F(z)}\right\} \tag{2.5}$$

$$= (c+be^{i\alpha})G(z).$$

By (2.3), (2.5) takes the following form.

$$c-ae^{i\alpha} - 1 + (a+b)e^{i\alpha}\frac{zF'(z)}{F(z)} + \frac{zw'(z)}{1-w(z)} + \tag{2.6}$$

$$\frac{(2a\cos\alpha - 2m\cos\alpha - c+be^{-i\alpha})zw'(z)}{(c+be^{i\alpha}) + \left[2a\cos\alpha - 2m\cos\alpha - c+be^{-i\alpha}\right]w(z)} = c - 1 -$$

$$\sum_{n=1}^{M} \lambda_n e^{i\alpha} + be^{i\alpha}\frac{zf'(z)}{f(z)} + \sum_{n=1}^{M} \frac{\lambda_n e^{i\alpha} zf_n'(z)}{f_n(z)}.$$

Hence

$$b\mathrm{Re}\left\{\frac{ze^{i\alpha}f'(z)}{f(z)}\right\} = -a\cos\alpha + (a+b)\mathrm{Re}\left\{\frac{e^{i\alpha}zF'(z)}{F(z)}\right\} + \tag{2.7}$$

$$+\mathrm{Re}\left\{\frac{2(a+b-m)\cos\alpha\ zw'(z)}{(1-w(z))\{c+be^{i\alpha}+w(z)\left[2a\cos\alpha-2m\cos\alpha-c+be^{-i\alpha}\right]\}}\right\}+$$

$$+\sum_{n=1}^{M}\lambda_n\cos\alpha - \sum_{n=1}^{M}\lambda_n \mathrm{Re}\left\{\frac{e^{i\alpha}\ zf'_n(z)}{f_n(z)}\right\}.$$

It is clear that $F(z)$ is regular in D and possesses a simple zero at the origin. Hence, without loss of generality, we may assume that $w(z)$ defined by $F(z)$ in (2.3) is regular in D. Also, it follows that $w(0) = 0$. Thus, if we show that $|w(z)| < 1$ in D then, by subordination, it will follow that $F(z)$ is spiral like of order $(\frac{m}{a+b})$ whenever $0 \leq \frac{m}{a+b} < 1$. Suppose there existed a point z_1 in D at which $|w(z_1)| = 1$, then by Jack's lemma $z_1w(z_1) = K\, w'(z_1)$, $K \geq 1$. For this value of $z = z_1$, we find that (2.7) takes the following form

$$b\ \mathrm{Re}\left\{\frac{z_1f'(z_1)e^{i\alpha}}{f(z_1)} - \beta\cos\alpha\right\} \leq \tag{2.8}$$

$$\leq \frac{-K(a+b-m)(\mathrm{Re}\ c-a\cos\alpha+m\cos\alpha)\cos\ \alpha}{|c+be^{i\alpha}+w(z)\{2a\cos\alpha-2m\cos\alpha-c+be^{-i\alpha}\}|^2} \leq 0$$

if $a+b-m \geq 0$ and $\mathrm{Re}\ c \geq (a-m)\cos\alpha$.

But, this contradicts that $f \in S(\alpha,\beta)$. Hence, we must have $|w(z)| < 1$ in D. This completes the proof of the theorem.

Some consequences of Theorem 1, are the following.

(A) Let $\beta_1 = \frac{1}{2}$, $\lambda_1 = \gamma$, $\lambda_m = 0$, for $m \geq 2$, $b = \delta$, $\alpha = 0$, $a+b = \beta^*$, $c+b = C$,

$f_1 = g \in S^*_{\frac{1}{2}=\beta_1}$ $(=K_o)$, and $f \in S^*_0$ then by Theorem 1, it follows that

$$F(z) = \left[\frac{C}{z^{C-\beta^*}}\int_0^z t^{C-1}\left(\frac{f(t)}{t}\right)^{\delta}\left(\frac{g(t)}{t}\right)^{\gamma}dt\right]^{1/\beta^*} \in S^*\left(0, \frac{2\beta^*-\gamma-2\delta}{2\beta^*}\right) \text{ for } \min(\operatorname{Re} C, a) \geq \delta + \frac{\gamma}{2}.$$

This is more general and sharper than the main theorem of Causey and White ([3], Thm 3.1, p. 459).

(B) Take $b = \xi_1, \rho = \lambda_1$, $\gamma = c-a$, $a+b = \beta^*$, $\alpha = 0 = \beta$, $M = 1$, $c+b = \xi+\delta = \beta^* + \gamma$, $f_1 = g$, $\beta_1 = \frac{1}{2}$ and $c \in R^+$. Then, Theorem B, follows with $F \in S^*(0, \frac{2\beta^*-2\xi-\rho}{2\beta^*})$ and $0 \leq \frac{\rho}{2} \leq \min(\delta, \delta-\gamma)$ if $\gamma > 0$ and $0 \leq \frac{\rho}{2} \leq \delta$ if $\gamma \leq 0$. This sharpens theorem 6 of ([7], p. 165).

(C) If we take $b = \xi$, $\lambda_1 = \rho$, $\gamma = c-a$, $a+b = \beta^*, \alpha = 0$, $M = 1$, $c+b=\xi+\delta=\beta^*+\gamma$, $f_1 = g$, $\beta = \frac{1}{2} = \beta_1$, $c \in R$, f, $g \in K_{\beta=0} \subset S^*_{1/2}$, then F of theorem $1 \varepsilon S^*_{\frac{2\beta^*-2\xi-\rho}{2\beta^*}}$ whenever $-\xi \leq \rho \leq \min(2\delta+\xi, 2\delta+\xi - 2\gamma)$. This is a slightly improved version of theorem 7 of [7] which implies that $F \in S^*_0$ and where $0 \leq \rho \leq \min\{2\delta+\xi, 2\delta+\xi-2\gamma + \frac{1}{2}\min(\frac{\beta^*}{\gamma}, \frac{\gamma}{\beta^*})$ if $\gamma > 0$ and $0 \leq \rho \leq 2\delta+\xi$ if $\gamma \leq 0$.

(D) If we let $b = 0 = \alpha$, $a = 1$, $c = 1 = M$, $f_1 \in S^*_{1/2} = S(0, \frac{1}{2}) \subsetneq K_0$ then by theorem 1, it follows that

$$F(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\lambda_1} dt \in S(0, 1 - \frac{\lambda_1}{2}) \text{ if } \frac{\lambda_1}{2} \leq 1.$$

This is clearly an improvement of Merkes-Wright's result [8] and also to that of Miller, Mocanu and Reade ([7], p. 165).

(E) If we let $a = \lambda_n = 0$, $b > 0$ then it follows that

$$F(z) = \left[\frac{c+be^{i\alpha}}{z^c}\int_0^z t^{c-1}\left[f(t)\right]^{be^{i\alpha}} dt\right]^{\frac{1}{be^{i\alpha}}} \quad \varepsilon \ S(\alpha,\beta) \quad \text{for} \quad f \ \varepsilon \ S(\alpha,\beta)$$

and $\min(\mathrm{Re}\ c,\ 0) > -b\beta$. If further $\alpha = 0 = \beta$ then $f \ \varepsilon \ S^*$ implies $F \ \varepsilon \ S^*$ for $\mathrm{Re}\ c > 0$. Whenerver c is real, this result is due to Singh [13].

(F) If we let $a = 0 = \lambda_n$ then the function F defined in (E) enables us to generalize Theorem 4 of [7]. Also, it is worth noticing that the function F defined in (E) is a generalization of α-spiral-β-convex function [12] which in turn is a generalization of alpha convex functions due to Mocanu [9].

A slightly different method than used for proof of Theorem 1, permits us to generalize Theorem E as follows.

THEOREM 2: - Let $\psi(z) = 1+\ldots$, and $\phi(z) = 1+\ldots$ be analytic in D with $\phi(z)\cdot\psi(z) \neq 0$ there. Let $\alpha \geq 0$, $\beta > 0$, $\mathrm{Re}(\gamma-\delta) = \alpha-\beta$, $\mathrm{Re}\ \gamma \geq 0$ and $f \ \varepsilon \ S^*$ then $F \ \varepsilon \ S^*_{m_0}$ for some m_o where

$$F(z) = \left[\left(\frac{\beta+\gamma}{z^{\gamma}\psi(z)}\right)\int_0^z f^{\alpha}(t)\phi(t)t^{\delta-1}dt\right]^{1/\beta} = z + \ldots, \tag{2.9}$$

and $0 \leq m_o < 1$. Furthermore, if $m = \max\limits_{z\varepsilon D} \frac{1}{\beta}\left\{\mathrm{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right)\right\} + m_o$

then m satisfies the following inequality

$$\left[\alpha + m\beta-\beta-\mathrm{Re}\left\{\frac{z\phi'(z)}{\phi(z)}\right\}\right]\left[|\gamma+\beta| + |\beta-2m\beta-\gamma|\right]^2 \leq \tag{2.10}$$

$$2\beta(1-m)(m\beta+\mathrm{Re}\{\gamma\}).$$

Moreover, if $G(z) = F(z)\psi^{1/\beta}(z)$, then $G \ \varepsilon \ S^*_m$ where m is the largest value obtained from (2.10).

PROOF OF THEOREM 2: - Let us write $G(z) = F(z)\psi^{1/\beta}(z)$ and $\frac{zG'(z)}{G(z)} = \frac{1+(1-2m)w(z)}{1-w(z)}$; $0 \leq m < 1$. Then, $w(z)$ is regular in D, $w(0) = 0$. Some easy simplifications give us

$$\alpha \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}(\gamma-\delta) + m\beta + (1-m)\operatorname{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\}\beta \tag{2.11}$$

$$+ \operatorname{Re}\left\{\frac{2\beta(1-m)zw'(z)}{(1-w(z))((\gamma+\beta)+(\beta-2m\beta-\gamma)w(z))}\right\} - \operatorname{Re}\left\{\frac{z\phi'(z)}{\phi(z)}\right\}.$$

Now, if it is not possible for all $z \in D$ that $|w(z)| < 1$ then there exists a $z_1 \in D$ for which $|w(z_1)| = 1$. Then by Jack's lemma $z_1 w'(z_1) = Kw(z_1)$, $K \geq 1$. For such a z_1, we have

$$\alpha \operatorname{Re}\left\{\frac{z_1 f'(z_1)}{f(z_1)}\right\} \leq \operatorname{Re}\{\gamma-\delta\} - \operatorname{Re}\left\{\frac{z\phi'(z)}{\phi(z)}\right\} + m\beta - \tag{2.12}$$

$$\frac{2\beta(1-m)(m\beta+\operatorname{Re}(\gamma))}{\{|(\gamma+\beta)| + |(\beta-2m\beta-\gamma)|\}^2}.$$

The last inequality leads to a contradiction on $f \in S^*$ whenever (2.10) is satisfied. Thus, we must have $|w(z)| < 1$ in D and the theorem is proved for $G \in S_m^*$ and it also follows that $F \in S_{m_0}^*$. A special case is of interest.

(A') Let $\operatorname{Im}(\gamma-\delta) = 0$. Then $\gamma-\delta = \alpha-\beta$. In this case (2.10) possesses simple solutions. There are possibilities (i) $\beta-2m\beta-\gamma \geq 0$, (ii) $\beta-2m\beta-\gamma \leq 0$. For (i), we obtain m must satisfy $m\beta+\gamma \geq 2\beta\left[\alpha-m\beta-\beta-\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right)\right](1-m)$ whereas in the case (ii) it must satisfy the inequality $\beta(1-m) \geq 2(\gamma+m\beta)\left[\alpha-\beta-m\beta - \operatorname{Re}\left\{\frac{z\phi'(z)}{\phi(z)}\right\}\right]$.

3. OTHER FUNCTIONS DEFINED BY INTEGRAL REPRESENTATIONS

THEOREM 3: - Let f be starlike and $F(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt$. Then $F \in S_m^*$

where m satisfies the inequality $m[|1+c|+|1-c-2m|]^2 \leq 2(1-m)(m+\mathrm{Re}\{c\})$ and $0\leq m<1$.

PROOF: - A simple calculation shows that

$$\left[c + \frac{zF'(z)}{F(z)}\right]F(z) = (1+c)f(z). \tag{3.1}$$

If possible, let us write

$$\frac{zF'(z)}{F(z)} = (1-m)\left(\frac{1+w(z)}{1-w(z)}\right) + m; \quad 0 \leq m < 1. \tag{3.2}$$

Then (3.1) and (3.2) gives us,

$$\frac{zf'(z)}{f(z)} = m + (1-m)\left(\frac{1+w(z)}{1-w(z)}\right) \tag{3.3}$$

$$+ \frac{2(1-m)\ zw'(z)}{[1-w(z)][(1+c)+(1-c-2m)w(z)]}.$$

From (3.2), it follows that $w(0) = 0$, $w(z)$ is regular in D, if it were not possible that $|w(z)| < 1$ in D then there exists $z_1 \varepsilon$ D such that $|w(z_1)| = 1$ and by Jack's lemma $z_1w'(z_1) = Kw(z_1)$ for some $K \geq 1$. Hence at $z = z_1$, by (3.3), we get

$$\mathrm{Re}\left\{\frac{z_1f'(z_1)}{f(z_1)}\right\} = m + \frac{2(1-m)K\mathrm{Re}\{(w(z_1)-1)((1+\bar{c})+(1-\bar{c}-2m)\bar{w}(z_1)\}}{|(1-w(z_1))|^2|(1+c)+(1-c-2m)w(z)|^2}$$

$$= m - \frac{2K(1-m)(m+c_1)}{|(1+c)+(1-c-2m)w(z_1)|^2}; \quad c_1 = \mathrm{Re}\{c\}$$

$$\leq m - \frac{2(1-m)(m+c_1)}{|(1+c)+(1-c-2m)w(z_1)|^2}; \quad c_1 \geq -m$$

$$\leq m - \frac{2(1-m)(m+c_1)}{(|1+c|+|1-c-2m|)^2}; \quad c_1 \geq -m.$$

But, then right hand side is ≤ 0. Hence, we have a contradiction for $f \varepsilon S_0^*$. Thus, we must have $|w(z)| < 1$ and Theorem 3 is proved.

As a consequence if we let c=1, then $F(z) = \frac{2}{z}\int_0^z f(t)dt$, $f \in S^*$, is starlike of order m where m is the largest possible positive value satisfying the inequality $2\left[\frac{\sqrt{17}-3}{4} - m\right]\left[m + \frac{\sqrt{17}+3}{4}\right] \geq 0$. Hence, we must have $m = \frac{\sqrt{17}-3}{4}$. Thus, we have recovered by our method, a result of Miller, Mocanu and Reade ([7], pp. 162-163), an improvement of a theorem of Libera [6]. Similarly, if we admit c to be real number such that $c+m \geq 0$ and m is the largest positive value satisfying the inequality $m \leq 2(1-m)(m+c)/\{|1+c|+|c+2m-1|\}^2$ then by Theorem 3, we obtain that for $f \in S^*$ the function $F(z) = (1+c)/z^c)\int_0^z t^{c-1}f(t)dt \in S_m^*$. It is easy to see that $m \geq \frac{1-c}{2}$ and hence, we must have $m = \{-(2c+1)+\sqrt{(2c-1)^2+8(1+c)}\}/4$. This is another, result of Miller, Mocanu and Reade ([7], pp. 162-163) obtained by our method.

THEOREM 4: - If $f \in S^*$, $\alpha \geq 0$ and $F(z) = \left[z^{\beta-1}\int_0^z [f(t)/t]^\alpha dt\right]^{1/\beta}$, $0 \leq \beta \leq 1$ then $F \in S_m^*$ where m satisfies the inequality

$$(\alpha-\beta+m\beta)\left[1+|2\beta-2m\beta-1|\right]^2 \leq 2\beta(1-m)\left[1-\beta(1-m)\right]. \tag{3.4}$$

PROOF: - If possible, let $\frac{zF'(z)}{F(z)} = (1-m)(\frac{1+w(z)}{1-w(z)}) + m = \frac{1+(1-2m)w(z)}{1-w(z)}$, $0 \leq m < 1$. Then, we obtain

$$\alpha\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \alpha-\beta+m\beta+\beta(1-m)\mathrm{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\} + \mathrm{Re}\left\{\frac{2\beta zw'(z)(1-m)}{[1-w(z)][1+(2\beta-2m\beta-1)w(z)]}\right\}. \tag{3.5}$$

Clearly, we have $w(0) = 0$, $w(z)$ is regular in D and if it is not possible $|w(z)| < 1$ in D then there exists a $z_1 \in D$ such that $|w(z_1)| = 1$ and by Jack's lemma $z_1w'(z_1) = Kw(z_1)$; $K \geq 1$. Then, for this value of z_1, we get

$$\alpha\mathrm{Re}\left\{\frac{z_1 f'(z_1)}{f(z_1)}\right\} \le \alpha-\beta+m\beta - \frac{2\beta(1-m)(1+m\beta-\beta)}{\{1+|2\beta-2m\beta-1|\}^2}. \tag{3.6}$$

This contradicts that $f \in S^*$. Hence, we must have $|w(z)| < 1$ and so proof is finished. This improves Theorem 3 of [7].

THEOREM 5: - If $f \in S^*$ then $F(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\alpha dt \in S_m^*$ for $0 \le \alpha \le 1$ and $m = \frac{(1-2\alpha)+\sqrt{4\alpha^2-4\alpha+9}}{4}$.

PROOF: - Choosing $\beta = 1$, in Theorem 4, we obtain $F \in S_m^*$ if m satisfies the inequality

$$\alpha - 1 + m - \frac{2m(1-m)}{[1+|1-2m|]^2} \le 0. \tag{3.7}$$

Now, there are two cases: (i) $m \ge \frac{1}{2}$ or (ii) $m \le \frac{1}{2}$. If $0 \le \alpha \le 1$ then (3.7) is always satisfied for $m \le \frac{1}{2}$. Hence, $m \ge \frac{1}{2}$. In this case we find, whenever

$$x_1 = \frac{(1-2\alpha)-\sqrt{4\alpha^2-4\alpha+9}}{4} \le m \le \frac{(1-2\alpha) + \sqrt{4\alpha^2-4\alpha+9}}{4} = x_2 \tag{3.8}$$

the inequality (3.7) is always satisfied. But, it is also true that $x_1 \le \frac{1}{2} \le x_2$. Hence, $m = \left[(1-2\alpha) + \sqrt{4\alpha^2 - 4\alpha + 9}\right]/4$.

Finally when $\alpha > 1$ the same method gives us the following improvement of a result of Silvia [12].

THEOREM 6: - Let $f \in S$, $f(z)f'(z)/z \ne 0$, $\frac{-\pi}{2} \le \lambda \le \frac{\pi}{2}$ and $\mathrm{Re}\{(e^{i\lambda} - \alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + \frac{zf''(z)}{f'(z)})\} > \beta\cos\lambda$, $0 \le \lambda \le 1$, $-\infty < \alpha < \infty$. Then f is λ-spiral like of order $m(>\beta)$, i.e. $f \in S(\lambda,m)$ if $\alpha \ge 1$ and $f \in S(\lambda,\beta)$ if $\alpha \le 1$ where m is the largest real number satisfying the inequality

$$(m-\beta)\{1+[1-4m(1-m)\cos^2\lambda]^{1/2}\} \leq 2m\alpha(1-m)\cos\lambda.$$

In special, if we take $\lambda = 0 = \beta$, then the class of alpha convex functions introduced by Mocanu [9] are produced if, further, $\alpha \geq 1$ then these functions are of order $m = \frac{\sqrt{8\alpha+\alpha^2}-\alpha}{4}$. This is a slight improvement of a result of Miller, Mocanu and Reade [7]. We omit the proof of this theorem, which is long but straightforward.

4. GENERALIZATION OF THEOREM B

First of all we note that $f(z) = z \in S^*$ and so trivially $S(z) = \frac{2}{z}\int_0^z t\,dt = z \in S^*$ but $T(z) = [z - \frac{2}{z}\int_0^z t\,dt]^{1/2} \equiv 0$, a non-univalent function. Hence, there is no chance of an exact analogue of theorem B. However, there are some variant analogues of theorem B.

THEOREM 7: - If $f \in K_0$ then $T(z) = 2[f(z) - \frac{1}{z}\int_0^z f(t)dt] \in S^*_m$ where $m = (\frac{\sqrt{17}-3}{4})$.

The proof of this theorem is similar to that of Theorem 3 and just follows by assuming that $0 \leq m \leq (1-m)/2(1+m)$ or by using theorem 3. This is an improvement of a result of Libera ([6], p. 757).

Theorem 7 suggests the following.

THEOREM 8: - If $f \in K_0$ then $F(z) = (\frac{e^{i\alpha} - e^{i\beta}}{z})\left[\int_0^z f(te^{i\alpha})dt - \int_0^z f(te^{i\beta})dt\right] \in S^*_m$ for $\alpha \neq \beta$ and $\alpha, \beta \in [0, 2\pi)$ where $m = (\sqrt{17}-3)/4$.

The proof of this theorem depends on the following lemma:

LEMMA 2: If $f \in K_\delta$ then $\alpha, \beta \in [0, 2\pi)$, $\alpha \neq \beta$, the function

$$F_o(z) = \left\{\frac{f(e^{i\alpha}z) - f(ze^{i\beta})}{e^{i\alpha} - e^{i\beta}}\right\} \in S^*_\delta.$$

The proof of this lemma is contained in ([1], p. 68), hence, we omit it.

Lemma 2 and Theorem 3 imply that F(z) defined in Theorem 8 belongs to S^*_m where m satisfies the inequality $m \leq (1-m)/2(1+m)$. This proves Theorem 8.

Similarly Lemma 2 and our Theorem 3 imply that F(z) defined by Theorem 1, of ([1], $f \in K_o$) belongs to S^*_m where m is given by the inequality of Theorem 3.

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