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ON HOLOMORPHIC FUNCTIONS WITH CERTAIN EXTREMAL PROPERTIES OF ITS ABSOLUTE VALUES

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ABSTRACT. This paper is concerned with a special class of holomorphic functions with extremal properties of its absolute values on arbitrary closed line segments in the complex plane. The main result is a geometrical characterization of the functions $z \to e^{az+b}$, $z \to (az+b)^n$ and $z \to (az+b)^{\alpha+i\beta}$ with $a,b \in \mathbb{C}$, $\alpha,\beta \in \mathbb{R}$, $n \in \mathbb{Z}$.

KEY WORDS AND PHRASES. Maximum respectively minimum of the absolute value $|f|$ is taken on at one of the endpoints of every closed line segment.

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1. INTRODUCTION.

The present work is closely related to the following problem raised by Rubel [1]: Find all entire functions f such that for every closed line segment L in the complex plane, wherever located an in whatever direction, the maximum of $|f|$ on L is taken on at one of the two endpoints of L .

As a secondary result of the solution for this problem we will obtain a simple characterization of the entire function $z \to e^{az+b}$. Suppose $f : G \to \mathbb{C}$ is a complex function holomorphic in the region G .

Then for all $z=x+iy\in G$ with $f(z) \neq 0$ the partial derivatives of first and second order of the function $w(x,y) = |f(z)|$ are given by [2]:

$$w_x = |f(z)|\operatorname{Re}\left(\frac{f'(z)}{f(z)}\right) , \quad w_y = -|f(z)|\operatorname{Im}\left(\frac{f'(z)}{f(z)}\right) ,$$

$$w_{xx} = |f(z)|\left\{\operatorname{Im}^2\left(\frac{f'(z)}{f(z)}\right) + \operatorname{Re}\left(\frac{f''(z)}{f(z)}\right)\right\} , \tag{1.1}$$

$$w_{yy} = |f(z)|\left\{\operatorname{Re}^2\left(\frac{f'(z)}{f(z)}\right) - \operatorname{Re}\left(\frac{f''(z)}{f(z)}\right)\right\} ,$$

$$w_{xy} = |f(z)|\left\{\operatorname{Im}\left(\frac{f'(z)}{f(z)}\right)\operatorname{Re}\left(\frac{f'(z)}{f(z)}\right) - \operatorname{Im}\left(\frac{f''(z)}{f(z)}\right)\right\} .$$

Moreover the formula of Taylor implies:

$$\begin{aligned} w(x+h,y+k) &= w(x,y) + hw_x(x,y) + kw_y(x,y) \\ &+ \frac{1}{2}\left\{h^2w_{xx}(x,y) + 2hkw_{xy}(x,y) + k^2w_{yy}(x,y)\right\} \\ &+ o(h^2+k^2) . \end{aligned} \tag{1.2}$$

Introducing the variable $\zeta := h+ik$ we deduce from (1.1) and (1.2):

$$\begin{aligned} |f(z+\zeta)| - |f(z)| = |f(z)|\left\{\operatorname{Re}\left(\frac{f'(z)}{f(z)}\zeta\right) + \frac{1}{2}\operatorname{Im}^2\left(\frac{f'(z)}{f(z)}\zeta\right) + \frac{1}{2}\operatorname{Re}\left(\frac{f''(z)}{f(z)}\zeta^2\right)\right\} \\ + o(|\zeta|^2) . \end{aligned} \tag{1.3}$$

By means of this equation we prove the following Lemma.

LEMMA 1. Let $f:G \to \mathbb{C}$ be holomorphic in the region G, $z \in G$ with $f(z) \neq 0$, $f'(z) \neq 0$ and

$$\operatorname{Re}\left(\frac{f(z)f''(z)}{f'^2(z)}\right) > 1 \quad \text{respectively} \quad \operatorname{Re}\left(\frac{f(z)f''(z)}{f'^2(z)}\right) < 1 .$$

Then there exists a line segment L through z such that $|f|$ does not reach its maximum respectively minimum at one of the endpoints of

L .

PROOF. Suppose $z \in G$ with $f(z) \neq 0$, $f'(z) \neq 0$

and $\mathrm{Re}\left(\frac{f(z)f''(z)}{f'^2(z)}\right) > 1$.

For real t in a sufficiently small neighbourhood of zero, we define $\zeta := i\,\frac{f(z)}{f'(z)}\,t$.

Then we have:

$$\mathrm{Re}\left(\frac{f'(z)}{f(z)}\zeta\right) = 0 \ , \ \mathrm{Im}\left(\frac{f'(z)}{f(z)}\zeta\right) = t \ ,$$

$$\mathrm{Re}\left(\frac{f''(z)}{f(z)}\zeta^2\right) = -\,t^2\,\mathrm{Re}\left(\frac{f(z)f''(z)}{f'^2(z)}\right) \ .$$

From these equations it follows by means of (1.3)

$$\left|f\left(z+i\,\frac{f(z)}{f'(z)}t\right)\right| - |f(z)| = \frac{t^2}{2}\,|f(z)|\left(1-\mathrm{Re}\left(\frac{f(z)f''(z)}{f'^2(z)}\right)\right) + o(t^2) \ . \qquad (1.4)$$

Hence there exists a $t_o \in R$ such sthat:

$$\left|f\left(z+i\,\frac{f(z)}{f'(z)}t_o\right)\right| < |f(z)| \quad \text{and}$$

$$\left|f\left(z-i\,\frac{f(z)}{f'(z)}t_o\right)\right| < |f(z)| \ .$$

The case $\mathrm{Re}\left(\frac{f(z)f''(z)}{f'^2(z)}\right) < 1$ is treated in the same way.

The next Lemma is an immediate consequence of the well known theorem of Picard [3], "Let g be a meromorphic function in the whole complex plane. If there exist three different numbers not belonging to the range of g , then g is constant".

LEMMA 2. Let f be a meromorphic function in the whole complex plane, which is not constant.

Then the function $g := \frac{ff''}{f'^2}$ is either a constant or there exist $z_o, z_1 \in \mathbb{C}$ with

$$\mathrm{Re}(g(z_o)) > 1 \quad \text{and} \quad \mathrm{Re}(g(z_1)) < 1 .$$

PROOF. Since f is meromorphic in all of $\mathbb{C}$ and not constant, also the function $g = \frac{ff''}{f'^2}$ is meromorphic in all of $\mathbb{C}$.
Then our Lemma immediately follows from the theorem of Picard.
Collecting the results obtained so far we end up with the following theorem:

THEOREM 1. Suppose f is a non constant function meromorphic in the whole complex plane such that also $g = \frac{ff''}{f'^2}$ is not a constant.
Then there exist two line segments L_o and L_1 such that neither the maximum of $|f|$ on L_o nor the minimum of $|f|$ on L_1 is taken on at the endpoints of these segments.
Next we consider the case that the expression $\frac{ff''}{f'^2}$ is a constant on $\mathbb{C}$.

LEMMA 3. Let $c = \gamma+i\delta$ be an arbitrary complex number. Then the solutions of the differential equation

$$\frac{ff''}{f'^2} = c \tag{1.5}$$

are giben by:

$$f(z) = \begin{cases} e^{az+b} & \text{for} \quad c = 1 , \\ (az+b)^{\frac{1}{1-c}} & \text{for} \quad c \neq 1 . \end{cases} \tag{1.6}$$

PROOF. Rewriting the differential equation (1.5) in the form

$$\frac{f''}{f'} = c\,\frac{f'}{f} , \tag{1.7}$$

it may easily be integrated [4].
The result is (1.6) with $a \in \mathbb{C}\setminus\{o\}$ and $b \in \mathbb{C}$.
In the case $c = \gamma+i\delta \neq 1$ the introduction of new variables α and β by $\alpha + i\beta := \frac{1}{1-c}$ leads to the relations

$$\alpha = \frac{1-\gamma}{(1-\gamma)^2+\delta^2} , \ \beta = \frac{\delta}{(1-\gamma)^2+\delta^2} \tag{1.8}$$

and thereby

$$\alpha = 0 \Leftrightarrow \gamma = 1 \ , \ \alpha > 0 \Leftrightarrow \gamma < 1 \ , \ \alpha < 0 \Leftrightarrow 1 < \gamma \ , \tag{1.9}$$

which will be needed later on.

The investigation of the functions $f(z) = e^z$ respectively $f(z) = z$ with regard to the extremal properties of their absolute values causes no difficulties. Since the simple similarity transformation $z \to az+b$ maps line segments into line segments there directly follows:

THEOREM 2. If f is a non-constant, entire function on $\mathbb{C}$ such that on every line segment L its absolute value $|f|$ takes on its maximum at one of the endpoints of L, then f is given either by $f(z) = e^{az+b}$ or by $f(z) = (az+b)^n$, $n \in \mathbb{N}$.

Theorem 2 completely solves the problem of Rubel mentioned at the beginning. In view of the equation $|\frac{1}{f(z)}| = \frac{1}{|f(z)|}$, a further consequence of Theorem 2 is:

THEOREM 3. If f is a non constant function meromorphic in the entire complex plane such that on every line segment L its absolute value $|f|$ reaches its minimum at one of the endpoints of L, then f is given either by $f(z) = e^{az+b}$ or by $f(z) = \frac{1}{(az+b)^n}$, $n \in \mathbb{N}$.

The combination of theorem 2 and Theorem 3 leads to a simple characterization of the exponential function:

THEOREM 4. Let f be a non-constant entire function such that on every line segment L the absolute value $|f|$ reaches its maximum as well as its minimum at the endpoints of L, then f is an exponential function of the form $f(z) = e^{az+b}$.

In view of Lemma 3 it seems to be interesting to investigate the general power function $f(z) = z^{\alpha+i\beta}$, $\alpha+i\beta \notin Z$, with regard to the extremal properties of its absolute value in the region

$$G := \{z \in \mathbb{C} \setminus \{o\} / -\pi < \arg z < \pi\} \tag{1.10}$$

Introducing polar coordinates $z = re^{i\varphi}$, the absolute value of f reads:

$$|f(z)| = r^{\alpha} \cdot e^{-\beta\varphi} . \tag{1.11}$$

On the half-lines with φ = const. the behaviour of $|f|$ is obvious. In the case of straight lines not running through the origin we have to consider separately those cutting the negative real axis. Finally in view of (1.11) it suffices to investigate $|f|$ on straight lines cutting the positive real axis vertically respectively on half-lines cutting the negative real axis vertically.

A straight line of the first kind is given in polar coordinate by:

$$r = \frac{p}{\cos\varphi} , \; p > 0 , \; -\frac{\pi}{2} < \varphi < \frac{\pi}{2} . \tag{1.12}$$

For $\alpha > 0$ it follows by means of elementary analysis that there exists exactly one minimum of $|f|$ on (1.12) given by:

$$\tan\varphi_1 = \frac{\beta}{\alpha} . \tag{1.13}$$

Similarly for $\alpha < 0$ there exists exactly one maximum of $|f|$ on (1.12) also fixed by (1.13). These results are in accordance with Lemma 1 and equation (1.9). Moreover the result (1.13) may be easily derived also via geometrical arguments by considering the geometry of the set of curves $r^{\alpha} \cdot e^{-\beta\varphi}$ = const.

For $1 > \alpha, \alpha \neq 0$ there occur two turning points, the position of which is fixed by:

$$\tan\varphi_{2,3} = \frac{\beta}{\alpha} \pm \frac{1}{\alpha}\sqrt{\frac{\alpha^2+\beta^2}{1-\alpha}} \quad . \tag{1.14}$$

On the half-lines mentioned above the arguments have to be slightly modified because of the limits:

$$\lim_{\arg z\to\pi} |f(z)| = r^\alpha \cdot e^{-\beta\pi} \; , \quad \lim_{\arg z\to-\pi} |f(z)| = r^\alpha \cdot e^{\beta\pi} \; .$$

Our final result reads:

THEOREM 5. Let G be the region defined by (1.10), K the class of all functions f holomorphic and non-constant in G with the further property that $g = \frac{ff''}{f'^2}$ is meromorphic in the entire complex plane. Every function $f \in K$ such that on any line segment $L \subset G$ its absolute value $|f|$ reaches its maximum (respectively minimum) in one of the endpoints of L is given by

$$f(z) = e^{az+b} \quad \text{or} \quad f(z)=(az+b)^{\alpha+i\beta} \quad \text{with} \quad \alpha \geq 0$$

$$(\text{respectively} \quad f(z) = e^{az+b} \quad \text{or} \quad f(z)=(az+b)^{\alpha+i\beta} \quad \text{with } \alpha \leq 0)$$

In passing it should be mentioned that. Ullrich [4] in his paper "Betragflächen mit ausgezeichnetem Krümmungsverhalten" ends up with the same functions which I have discussed in my paper [5], too.

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