Internat. J. Math. & Math. Sci.
Vol. 5 No. 1 (1982) 41-48

SMALLEST CUBIC AND QUARTIC GRAPHS WITH A GIVEN NUMBER OF CUTPOINTS AND BRIDGES

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(Received April 6, 1981)

ABSTRACT. For positive integers b and c, with c even, satisfying the inequalities $b + 1 \leq c \leq 2b$, the minimum order of a connected cubic graph with b bridges and c cutpoints is computed. Furthermore, the structure of all such smallest cubic graphs is determined. For each positive integer c, the minimum order of a quartic graph with c cutpoints is calculated. Moreover, the structure and number of all such smallest quartic graphs are determined.

KEY WORDS AND PHRASES. *Cutpoints, Bridges, Cubic graphs, Quartic graphs.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. *05C99*

1. INTRODUCTION.

Eitner and Harary [1] showed that if G is a connected cubic graph with $b(\geq 1)$ bridges and $c(\geq 1)$ cutpoints, then c is even and $b + 1 \leq c \leq 2b$. They further showed that for positive integers b and c, with c even, satisfying $b + 1 \leq c \leq 2b$, there exists a connected cubic graph with b bridges and c cutpoints. We determine the smallest order of such cubic graphs as well as their structure.

Graphs having only vertices of even degree do not contain bridges (their components are eulerian). Thus every quartic (4-regular) graph is bridgeless. For every positive integer c, the minimum order of a quartic graph with c cutpoints is computed. The structure and number of all such smallest quartic graphs are determined.

Throughout the paper we follow [2] and [3] for basic terminology in graph theory.

2. SMALLEST CUBIC GRAPHS WITH A GIVEN NUMBER OF CUTPOINTS AND BRIDGES.

THEOREM 2.1. Let b and c be positive integers, with c even, such that $b + 1 \leq c \leq 2b$. Then the minimum order of a connected cubic graph with b bridges and c cutpoints is $2b + c + 6$.

PROOF. Among the connected cubic graphs having b bridges and c cutpoints, let G be one of minimum order p. Let k ($0 \leq k \leq c$) be the number of cutpoints incident with exactly one bridge; then, of course, G has $c - k$ cutpoints incident with three bridges.

Let n be the number of blocks in G. Also, for $v \in V(G)$, denote the number of blocks of G containing v by n(v) (the number n(v) is sometimes called the block index of v). Thus, $n(v) = 1$ if v is not a cutpoint, $n(v) = 2$ if v is a cutpoint incident with exactly one bridge and $n(v) = 3$ if v is a cutpoint incident with three bridges.

Using the elementary formula of [4],

$$n = 1 + \sum_{v \in V(G)} [n(v) - 1],$$

we see that $n = 1 + 2k + 3(c - k) - c$ or

$$n = 2c - k + 1. \qquad (2.1)$$

If we sum the number of bridges incident with a cutpoint over all cutpoints, we obtain $2b = 3(c - k) + k$ so that

$$k = (3c - 2b)/2. \qquad (2.2)$$

Substituting the expression for k in (2.2) into (2.1), we obtain

$$n = 1 + b + c/2. \tag{2.3}$$

Thus, among the $1 + b + c/2$ blocks of G, the b bridges are acyclic and $1 + c/2$ are cyclic.

An end-block of G necessarily has one vertex of degree 2 and all others of degree 3. The smallest block with this property has order 5 and is the unique block B_1 of Figure 1. Necessarily, then, every end-block of G is isomorphic to B_1.

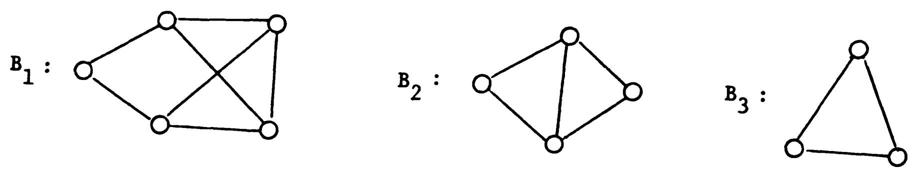

Figure 1

In a cubic graph the smallest cyclic block with exactly two cutpoints is isomorphic to the block B_2 of Figure 1 while the smallest block with exactly three cutpoints is isomorphic to the block B_3 of Figure 1. Indeed, any block of G having exactly i cutpoints ($i \geq 3$) must be isomorphic to the cycle C_i. Thus, if a cyclic block of G contains exactly i cutpoints, then the block is isomorphic to B_i if $i = 1,2$ and is isomorphic to C_i if $i \geq 3$. For $i \geq 1$, let n_i denote the number of cyclic blocks containing exactly i cutpoints of G. Thus,

$$n = b + \sum_{i=1} n_i. \tag{2.4}$$

We may assume that $n_3 = 0$; for if $n_3 > 0$, then G can be transformed into a cubic graph G* of order p having b bridges, c cutpoints and $n_3 = 0$. In order to see this, let B be a block of G having order 3 and let A be an end-block of G (see Figure 2). Let u be the cutpoint of G in A, and let G' be the graph obtained from G by deleting the edges of B and the non-cutpoints of A. Let H be the component of G' containing u, and let v be the vertex of H that belongs to B. Let w_1 and w_2 be the other two vertices of B, where $H_i (i = 1,2)$ is the component of G' containing w_i.

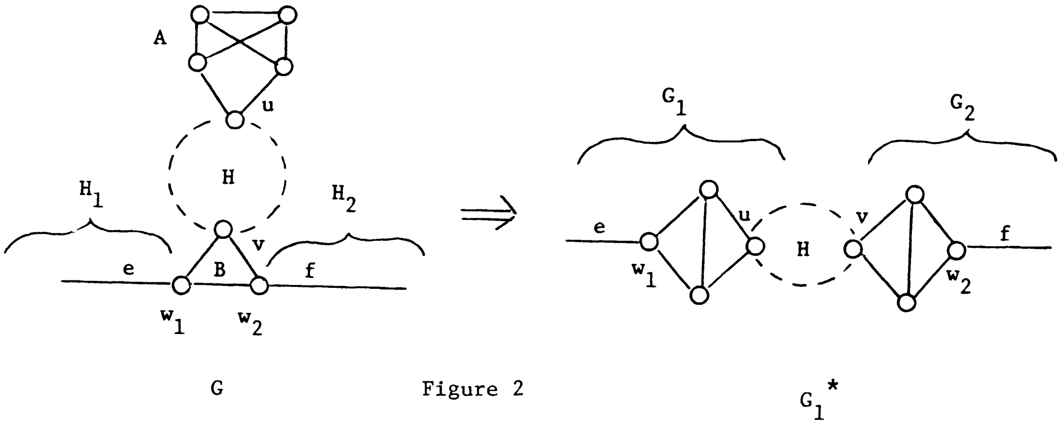

Figure 2

For i = 1,2, the graph G_i is defined by identifying a vertex of degree 2 of the block B_2 (of Figure 1) with the vertex w_i of H_i. The graph G_1^* is now obtained by identifying the vertex u of H with the vertex of degree 2 in G, and identifying v with the vertex of degree 2 in G_2 (see Figure 2). The cubic graph G_1^* has $n_3 - 1$ blocks isomorphic to C_3 and has order p, b bridges and c cutpoints. Continuing this procedure, if necessary, we arrive at the desired cubic graph G^*. Thus we can assume that $n_3 = 0$ for the graph G.

Next we show that $n_i = 0$ for all $i \geq 4$. Suppose that $n_j > 0$ for some $j \geq 4$. Then G contains a cycle C_j: $w_1, w_2, \ldots, w_j, w_1$, where then each $w_i (1 \leq i \leq j)$ is a cutpoint of G. Let F be the graph obtained by deleting the edges of C_j from G, and let $F_i (1 \leq i \leq j)$ be the component of F containing w_i.

Let $\hat{B}$ be an end-block of G that is also a block of F_1, and let w_o be the cutpoint of G belonging to $\hat{B}$. Note that $\hat{B}$ is isomorphic to the block B_1 of Figure 1. We construct the graph $\hat{G}$ by (1) deleting the edges $w_{j-1}w_j$ and w_jw_1 from G, (2) adding the edge w_1w_{j-1}, (3) deleting all vertices of $\hat{B}$ from G except w_o, (4) identifying one vertex of degree 2 in the block B_2 of Figure 1 with w_o, and (5) identifying the other vertex of degree 2 in B_2 with the vertex w_j of F_j. The graph $\hat{G}$ is cubic and has b bridges, c cutpoints and order p-2; however, this is impossible due to the manner in which G was chosen. Hence we may assume that $n_i = 0$ for $i \geq 3$.

The expression (2.4) for the number n of blocks of G can now be written as

$$n = b + n_1 + n_2. \tag{2.5}$$

It follows (by considering the block-cutpoint tree (see [5]) of G for example) that the number n_1 of end-blocks of G is given by

$$n_1 = 2 + (c - k) = 2 + b - c/2, \qquad (2.6)$$

where the latter expression for n_1 is a consequence of (2.2). By combining (2.3), (2.5) and (2.6), we have

$$n_2 = n - n_1 - b = (1 + b + c/2) - (2 + b - c/2) - b = c - 1 - b. \qquad (2.7)$$

Thus, (2.6) and (2.7) imply that the order p of G is

$$p = (c - k) + 5n_1 + 4n_2 = (b - c/2) + 5(2 + b - c/2) + 4(c - 1 - b) = 2b + c + 6.$$

By the preceding proof, it follows that each cyclic block of a connected cubic graph having b bridges, c cutpoints and order 2b + c + 6 is isomorphic to one of B_1, B_2 (of Figure 1) and C_3 and, conversely, if G is a connected cubic graph with b bridges and c cutpoints, every cyclic block of which is isomorphic to B_1, B_2 or C_3, then G has order 2b + c + 6. It therefore follows that the construction of Eitner and Harary [1] of a connected cubic graph with b bridges and c cutpoints for all b and c, with c even, and $b + 1 \le c \le 2b$ has, in fact, the minimum possible order, namely 2b + c + 6.

3. SMALLEST QUARTIC GRAPHS WITH A GIVEN NUMBER OF CUTPOINTS

THEOREM 3.1. For a positive integer c, the minimum order of a quartic graph with c cutpoints is (7c + 15)/2 if c is odd and (7c + 18)/2 if c is even.

PROOF. Among the quartic graphs with c cutpoints, let G be one of minimum order p. Suppose that G has n blocks, and let n(v) denote the number of blocks containing a vertex v of G. If v is not a cutpoint, then n(v) = 1; while if v is a cutpoint, then n(v) = 2. As mentioned in the proof of Theorem 2.1,

$$n = 1 + \sum_{v \in V(G)} [n(v) - 1].$$

Hence it follows that

$$n = 1 + c. \qquad (3.1)$$

Let n_i $(1 \le i \le c)$ be the number of blocks of G containing exactly i cutpoints. Thus

$$\sum_{i=1}^{\infty} i n_i = 2c. \qquad (3.2)$$

Every end-block of G is necessarily isomorphic to the block E_1 of Figure 3 and every block of G containing exactly two cutpoints is isomorphic to the block E_2 of Figure 3. Note that each of E_1 and E_2 has order 6. Moreover, each block of G containing exactly i cutpoints ($i \geq 3$) is isomorphic to the cycle C_i.

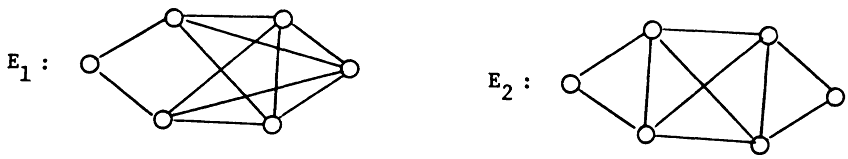

Figure 3

By an argument completely analogous to the proof of Theorem 2.1, it can be shown that $n_i = 0$ for all $i \geq 4$. Hence expressions (3.1) and (3.2) become

$$n_1 + n_2 + n_3 = 1 + c \tag{3.3}$$

and

$$n_1 + 2n_2 + 3n_3 = 2c, \tag{3.4}$$

respectively. Moreover, the order of G is

$$p = 6n_1 + 6n_2 + 3n_3 - c. \tag{3.5}$$

Eliminating n_1 in (3.3) and (3.4), we obtain

$$n_3 = (c - 1 - n_2)/2. \tag{3.6}$$

With the aid of (3.3) and (3.6), the expression for p in (3.5) becomes

$$p = (7c + 15 + 3n_2)/2. \tag{3.7}$$

Expression (3.7) implies that $p \geq (7c + 15)/2$. For c odd we construct a quartic graph G, with c cutpoints and order $(7c + 15)/2$, implying that $(7c + 15)/2$ is the minimum order of such a graph and that $n_2 = 0$ in such graphs. If $c = 1$, then G_1 is the graph of order 11 shown in Figure 4. Suppose that $c = 2k + 1 \geq 3$. Let $B_1, B_2, \ldots, B_k$ be k blocks isomorphic to C_3, where B_i and B_{i+1} ($1 \leq i \leq k - 1$) have a cutpoint in common. Also, every vertex belonging to exactly one B_j ($1 \leq j \leq k$)

also belongs to a block isomorphic to E_1. The graph so constructed is G_1. Thus G_1 has k blocks isomorphic to C_3, k + 2 blocks isomorphic to E_1, 2k + 1 (=c) cutpoints and order (7c + 15)/2.

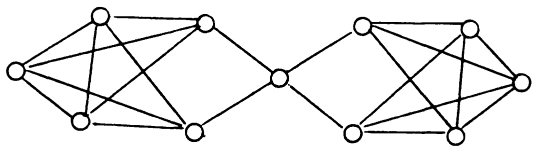

Figure 4

If c is even, then (3.7) implies that $n_2 \geq 1$ and $p \geq (7c + 18)/2$. We construct a quartic graph G_2 with c cutpoints and order (7c + 18)/2, thereby verifying that this is the minimum order and, further, that $n_2 = 1$. Let $c = 2m \geq 2$. If c = 2, take G_2 to be the graph of Figure 5. Otherwise, let G_2' be the graph having m - 1 blocks $B_1, B_2, \ldots, B_{m-1}$, each isomorphic to C_3, where B_i and B_{i+1} $(1 \leq i \leq m-2)$ have a cutpoint in common. Let w be a vertex of degree 2 in G_2'. Identify with w a vertex of degree 2 in a block isomorphic to E_2. The remaining vertex of degree 2 in E_2 is identified with the vertex of degree 2 in a block isomorphic to E_1. Every other vertex of degree 2 in G_2' is identified with the vertex of degree 2 in a block isomorphic to E_1. The resulting graph is G_2, which then has m - 1 blocks isomorphic to C_3, m + 1 blocks isomorphic to E_1, one block isomorphic to E_2, 2m (=c) cutpoints and order (7c + 18)/2.

The preceding proof also has the following corollary.

COROLLARY. For each odd positive integer c, there is exactly one quartic graph with c cutpoints and having order (7c + 15)/2. For each even positive integer c, there are exactly [c/4] quartic graphs with c cutpoints and order (7c + 18)/2 if $c \geq 4$ and exactly one such quartic graph if c = 2.

The problem of determining the number of (non-isomorphic) connected cubic graphs with b bridges, c cutpoints $(b + 1 \leq c \leq 2b)$ and order 2b + c + 6 remains unsettled.

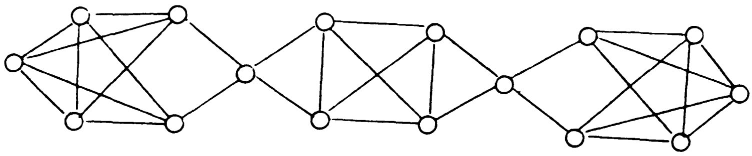

Figure 5

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