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ON SOME FIXED POINT THEOREMS IN BANACH SPACES

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ASTRACT. In this paper, some fixed point theorems are proved for multi-mappings as well as a pair of mappings. These extend certain known results due to Kirk, Browder, Kanna, Ćirić and Rhoades.

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1. INTRODUCTION.

A result of continuing interest in fixed point theory is one due to Kirk [6]. This states that a non-expansive self-mapping of bounded, closed and convex subset possessing normal structure in a reflexive Banach space has a fixed point. The interest in this result has been further enhanced due to simultaneous and independent appearance of results of Browder [2] and Göhde [5] which are essentially special cases of the result of Kirk. Recently Kannan [6] and Ćirić [3] have obtained results in basically the same spirit by suitably modifying the non-expansive condition on the mapping and the condition of normal structure on the underlying set. In this paper we give a fixed point result for multi-mappings (Theorem 2.1) and extend the results of Kannan [6] and Ciric [3] to a pair of mappings (Theorems 3.1 and 3.2). This enables us to establish convergence of Ishikawa iterates (cf. [9]) for a pair of mappings.

2. A FIXED POINT THEOREM FOR MULTI-MAPPINGS.

Let K be a closed, bounded and convex subset of a Banach space X. For $x \in X$, let $\delta(x;K)$ denote $\sup \{ ||x-k|| : k \in K\}$ and let $\delta(K)$ denote the diameter of K. Recall that a point $x \in K$ is called a non-diametral point of K if $\delta(x;K) < \delta(K)$ and that K is said to have normal structure whenever given any closed bounded convex subset C of K with more than one point, there exists a non-diametral $x \in C$. It is well-known (cf. [4]) that a compact convex subset of an arbitrary Banach space and a closed, bounded and convex subset of a uniformly convex Banach space have normal structure. With K as before, let $r(K)$ denote the radius of K : $\inf \{ \delta(x,K) : x \in K\}$ and let K_c denote the Chebyshev centre of K : $\{ x \in K : r(K) = \delta(x,K) \}$. It is well known (cf. Opial [8]) that if K is a non-empty weakly compact convex subset of a Banach space X, then K_c is nonempty closed convex subset of K and, furthermore if K has normal structure, then $\delta(K_c) < \delta(K)$ (whenever $\delta(K) > 0$). Let 2^K denote the collection of all non-empty subsets of K and, for $A,B \in 2^K$ let $\delta(A,B)$ denote $\sup \{ ||a-b|| : a \in A, b \in B\}$.

Theorem 2.1. Let K be a nonempty weakly compact convex subset of the Banach space X. Assume K has normal structure. Let $T:K \to 2^K$ be a mapping satisfying: for each closed convex subset F of K invariant under T, there exists some $\alpha(F)$, $0 \leq \alpha(F) < 1$, such that

$$\delta(Tx,Ty) \leq \max \{ \delta(x,F), \alpha(F)\, \delta(F) \}$$

for each $x, y \in F$.

Then T has a fixed point x_o satisfying $Tx_o = \{x_o\}$.

Proof. We imitate in parts the proof of Kirk's theorem. Let $\mathfrak{F}$ denote the collection of non-empty closed convex subsets C of K that are left invariant by T(i.e., $TC \subset C$, where $TC = \cup\{Tc : c \in C \}$). Order $\mathfrak{F}$ by set-inclusion. By weak compactness of K, we can apply Zorn's lemma to get a minimal element M. It suffices to show that M is a singleton. Suppose that M contains more than one element. By the definition of normal structure there exists $x_o \in M$ such that

$$\sup \{ ||x_o-y|| : y \in M \} = \delta(x_o,M) < \delta(M),$$

Hence $\delta(x_o,M) \leq \alpha_1(M)\, \delta(M)$ for some α_1, $0 < \alpha_1 < 1$.

If $\delta(Tx, Ty) \leq \delta(x,M)$ for all $x, y \in M$, let $M_\delta = \{x \in M: \delta(x,M) \leq \alpha_1 \delta(M)\}$.

Otherwise, by hypothesis there exists $\alpha(M)$, $0 \leq \alpha(M) < 1$, such that

$\delta(Tx, Ty) \leq \alpha\, \delta(M)$ for some $x, y \in M$.

Let $\beta = \max\{\alpha, \alpha_1\}$ and $M_\delta = \{x \in M: \delta(x,M) \leq \beta\delta(M)\}$.

As $x_o \in M_\delta$, M_δ is nonempty. Evidently, M_δ is convex. Since $x \to \delta(x,M)$ is continuous, M_δ is closed.

Let $x \in M_\delta$

$$\begin{aligned}\delta(Tx, Ty) &\leq \max\ \{\delta(x,M), \alpha\, \delta(M)\}\\ &\leq \beta\ \ \delta(M) \text{ for } y \in M.\end{aligned}$$

Hence $T(M)$ is contained in a closed ball of arbitrary centre in Tx and radius $\beta\delta(M)$. By the minimality of M, if $m \in Tx$, then $M \subset U(\ m : \beta\delta(M))$ (the closed ball of centre m and radius $\beta\delta(M)$), whence $m \in M_\delta$ and $T(M_\delta) \subset M_\delta$. But $\delta(M_\delta) \leq \beta\delta(M) < \delta(M)$ which contradicts the minimality of M. Thus M is a singleton and this completes the proof.

Corollary 2.2. Let K be a nonempty weakly compact convex subset of the Banach space X. Assume K has normal structure. Let T be a mapping of K into itself which satisfies: for each closed convex subset F of K invariant under T there exists some $\alpha(F)$, $0 \leq \alpha(F) < 1$, such that

$$||Tx-Ty|| \leq \max\{\ \delta(x,F),\ \alpha\delta(F)\ \}$$

for each $x, y \in F$. Then T has a fixed point.

Corollary 2.3. Let K be a nonempty weakly compact convex subset of the Banach space X. Assume K has normal structure. Let T be a mapping of K into itself which satisfies: for each closed convex subset F of K invariant under T there exists some $\alpha(F)$, $0 \leq \alpha(F) < 1$, such that

$$||Tx-Ty|| \leq \max\{\ ||x-y||,\ r(F),\ \alpha\, \delta(F)\ \}$$

for each $x, y \in F$. Then T has a fixed point.

Remark. The preceeding results generalize the results of Kirk [7] and Browder [2].

3. COMMON FIXED POINTS OF MAPPINGS.

Theorem 3.1. Let K be a weakly compact convex subset of the Banach space X. Let T_1, T_2 be two mappings of K into itself satisfying:

(1) $$||T_1x - T_2y|| \le \max \{ (||x-T_1x||+||y-T_2y||)/2, (||x-T_2y||+||y-T_1x||)/3, (||x-y||+||x-T_1x||+||y-T_2y||)/3\}$$

for each x, y ε K,

(2) $T_1C \subset C$ if and only if $T_2C \subset C$ for each closed subset C of K;

(3) either $\sup_{z \varepsilon C} ||z-T_1z|| \le \delta(C)/2$,

or $\sup_{z \varepsilon C} ||z-T_2z|| \le \delta(C)/2$

holds for each closed convex subset C of K invariant under T_1 and T_2.

Then there exists a unique common fixed point of T_1 and T_2.

Proof. Let $\mathfrak{F}$ denote the family of all non-empty closed convex subsets of K, each of which is mapped into itself by T_1 and T_2. Ordering $\mathfrak{F}$ by set-inclusion, by weak compactness of K and Zorn's lemma, we obtain a minimal element F of K. Without loss of generality, assume that

$$\sup_{z \varepsilon F} ||z-T_2z|| \le \delta(F)/2.$$

Let x ε F_c. Since $\delta(F)/2 \le r(f)$, we obtain using (1) that $||T_1x-T_2y|| \le r(F)$. (y ε F). This gives that $T_2(F) \subset U(T_1x : r(F)) = U$, whence $T_2(F \cap U) \subset F \cap U$ and by hypotheses (2) $T_1(F \cap U) \subset F \cap U$. By the minimality of F, we obtain $F \subset U$. This gives $\delta(T_1x,F) = r(F)$, whence $T_1x \;\varepsilon\; F_c$. Therefore, $T_1(F_c) \subset F_c$ and by hypothesis (2) $T_2(F_c) \subset F_c$. We now show that if F contains more than one element, then F_c is a proper subset of F. Assume the contrary that $F_c = F$. Since $\delta(x,F) = r(F)$ for each x ε F, we obtain $\delta(F) = r(F) = \delta(x,F)$, (x ε F). Again from (1), we get

$$\begin{aligned} ||T_1x - T_2y|| &\le \max \{3\, \delta(F)/4,\ (\delta(F) + \delta(F))/3, \\ &\qquad (\delta(F) + \delta(F) + \delta(F)/2)/3\} \\ &= 5\delta(F)/6. \end{aligned}$$

The same argument as before yields $\delta(T_1x,F) \leq 5\delta(F)/6 < \delta(F)$, which is a contradiction. Consequently, if F contains more than one element, then F_c is a proper subset of F. But this in view of above contradicts the minimality of F. Hence F contains exactly one element, say, x_0, whence $T_1x_0 = x_0 = T_2x_0$. Assume there exists another element $y_0 \in K$ such that $T_1y_0 = y_0 = T_2y_0$. Then using (1), we obtain

$$||T_1x_0 - T_2y_0|| \leq \frac{2}{3} ||T_1x_0 - T_2y_0|| ,$$

whence $$x_0 = T_1x_0 = T_2y_0 = y_0.$$

THEOREM 3.2. Let K be a weakly compact convex subset of the Banach space X. Assume K has normal structure. Let T_1, T_2 be mappings of K into itself satisfying:

(1) $$||T_1x - T_2y|| \leq \max \{(||x - T_1x|| + ||y-T_2y||)/2, \\ (||x-T_2y|| + ||y-T_1x||)/2, \\ (||x-y||+||x-T_1x|| + ||y-T_2y||)/3\}$$

for each $x,y \in K$,

(2) $T_1C \subset C$ if and only if $T_2C \subset C$ for each closed convex subset C of K,

(3) either $\sup_{z \in D} ||z - T_1z|| \leq r(D)$,

or $\sup_{z \in D} ||z - T_2z|| \leq r(D)$

holds for each closed convex subset D of K invariant under T_1 and T_2.

Then there exists a unique common fixed point of T_1 and T_2.

PROOF. Let $\mathfrak{F}$ be as in Theorem 3.1. Exactly as in Theorem 3.1., $\mathfrak{F}$ has a minimal element F. Without loss of generality, assume that $\sup_{z \in F} ||z-T_2z|| \leq r(F)$. Let $x \in F_c$. Then using (1) we obtain

$$||T_1x - T_2y|| \leq r(F). \quad (y \in F)$$

This gives exactly as in Theorem 3.1 that $T_1(F_c) \subset F_c$ and $T_2(F_c) \subset F_c$. Since K has normal structure, one has $\delta(F_c) < \delta(F)$ if K contains more than one element, which contradicts the minimality of F. Thus F contains precisely one element, which is the unique common fixed point of T_1 and T_2 as in Theorem 3.1.

REMARK. One can replace condition (1) of Theorem 3.2 by

(1) $$||T_1x - T_2y|| \leq \max \{||x-y||, (||x-T_1x|| + ||y-T_2y||)/2 , \\ (||x-T_2y|| + ||y-T_1x||)/3, (||x-y||+||x-T_1x||+||y-T_2y||)/3\}.$$

This also yields the existence of a common fixed point of T_1 and T_2. However, it need not be unique.

THEOREM 3.3. Let K be a weakly compact convex subset of the Banach space X. Assume K has normal structure. Let T_1, T_2 be mappings of K into itself satisfying (2) and (3) of the preceding theorem and,

$$(1) \quad ||T_1x - T_2y|| \leq \max \{||x-y||, ||x-T_1x||, ||x-T_1y||, ||x-T_2x||, ||x-T_2y||\}.$$

Then there exists a common fixed point of T_1 and T_2.

The proof of the above theorem is similar to that of Theorem 3.2 and hence it is omitted.

4. ISHIKAWA ITERATION FOR COMMON FIXED POINTS.

A uniformly convex Banach space is reflexive. A bounded, closed and convex subset of a uniformly convex Banach space is therefore weakly compact; also, it has normal structure. Hence Theorems 2.1, 3.2 and 3.3 can be particularized to such a setting. Rhoades [9] has extended a result of Ćirić (cf. [3], Theorem 2) to a wider class of transformations by using Ishikawa iterative scheme. With a suitable modification of arguments, this extends to a pair of mappings of the type as in Theorem 3.2.

THEOREM 4.1. Let K be a non-empty closed bounded and convex subset of a uniformly convex Banach space X. Let T_1, T_2 be mappings of K into itself satisfying (1), (2) and (3) of Theorem 3.2. Let the sequence $\{x_n\}$ of iterates be defined by

$$(4) \qquad x_0 \in K ,$$

$$(5) \qquad y_n = (1 - \beta_n)x_n + \beta_n T_1 x_n , \qquad n \geq 0 ,$$

$$(6) \qquad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_2 y_n , \qquad n \geq 0 ,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy (i) $0 \leq \alpha_n, \leq \beta_n \leq 1$ for all n, (ii) $\sum_n \alpha_n(1 - \alpha_n) = \infty$ and , (iii) $\overline{\lim}\, \beta_n = \beta < 1$. Then $\{x_n\}$ converges to the unique common fixed point of T_1 and T_2.

PROOF. The existence of the unique common fixed point of T_1 and T_2 results from Theorem 3.2. Let the unique common fixed point be v. From (1)

$$||T_1x_n - v|| \leq ||x_n - v||$$

and

$$||T_2x_n - v|| \le ||x_n - v|| .$$

Following exactly the same lines as in the proof of Theorem 1 of [9] we obtain subsequences y_{n_k}, x_{n_k} of y_n, x_n respectively such that

(7) $$\lim_k ||x_{n_k} - T_2\, y_{n_k}|| = 0$$

we show that

(8) $$\lim ||x_{n_k} - T_1\, x_{n_k}|| = 0 .$$

It would be sufficient, with (7), to show that $\lim_k ||T_1x_{n_k} - T_2y_{n_k}|| = 0.$

For any integer n, from

$$||T_1x_n - T_2y_n|| \le (||x_n - T_1x_n|| + ||y_n - T_2y_n||)/2 ,$$

we obtain

(9) $$||T_1x_n - T_2y_n|| \le (2 - \beta_n)||\, x_n - T_2y_n||/(1 - \beta_n).$$

It follows from

$$||T_1x_n - T_2y_n|| \le (||x_n - T_2y_n|| + ||y_n - T_1x_n||)/3 ,$$

that

(10) $$||T_1x_n - T_2y_n|| \le (2 - \beta_n)||x_n - T_2y_n||/\ (2 + \beta_n).$$

From

$$||T_1x_n - T_2y_n|| \le (||x_n - y_n|| + ||x_n - T_1x_n||+||y_n-T_2y_n||)/3$$

we obtain

(11) $$||T_1x_n - T_2y_n|| \le ||x_n - T_2y_n||\ /\ (1 - \beta_n) .$$

From (9) - (11) we obtain

$$||T_1x_n - T_2y_n|| \le 2||x_n - T_2y_n||\ /\ (1 - \beta_n) .$$

Therefore,

$$||T_1x_{n_k} - T_2y_{n_k}|| \le 2\ ||x_{n_k} - T_2y_{n_k}||/(1 - \beta_{n_k})$$

and (7) implies $\lim_k ||T_1x_{n_k} - T_2y_{n_k}|| = 0,$

whence

$$\lim_k ||x_{n_k} - T_1x_{n_k}|| = 0 ,$$

Now let us prove that this implies that

$$\lim_{k\to\infty} ||x_{n_k} - T_2x_{n_k}|| = 0.$$

This follows easily from

$$\begin{aligned} ||x_{n_k} - T_2x_{n_k}|| &\le ||x_{n_k} - T_1x_{n_k}|| + ||T_1x_{n_k} - T_2x_{n_k}|| \\ &\le ||x_{n_k} - T_1x_{n_k}|| + \max\{(||x_{n_k} - T_1x_{n_k}||+||x_{n_k}-T_2x_n||)/2, \\ &\quad (||x_{n_k} - T_2x_{n_k}|| + ||x_{n_k} - T_1x_{n_k}||)/3 , \\ &\quad (||x_{n_k} - x_{n_k}|| + ||x_{n_k} - T_1x_{n_k}|| + ||x_{n_k} - T_2x_{n_k}||)/3\}. \end{aligned}$$

which tends to 0 as $k \to \infty$ since

$$||x_{n_k} - T_1x_{n_k}|| \to 0 \text{ as } k \to \infty .$$

Also $\quad ||T_1x_{n_k} - T_1x_{n_\ell}|| \le ||T_1x_{n_k} - T_2x_{n_k}|| + ||T_2x_{n_k} - T_1x_{n_\ell}||$

From (1) of Theorem 3.2,

$$\begin{aligned} ||T_1x_{n_\ell} - T_2x_{n_k}|| \le \max\{&[||x_{n_\ell} - T_1x_{n_\ell}|| + ||x_{n_k} - T_2x_{n_k}||]/2 , \\ &[||x_{n_\ell} - T_2x_{n_k}|| + ||x_{n_k} - T_1x_{n_\ell}||]/3\}, \\ &[||x_{n_\ell} - x_{n_k}|| + ||x_{n_\ell} - T_1x_{n_\ell}|| + ||x_{n_k} - T_2x_{n_k}||]/3 . \end{aligned}$$

If

$$||T_1x_{n_\ell} - T_2x_{n_k}|| \le [||x_{n_\ell} - T_2x_{n_k}|| + ||x_{n_k} - T_1x_{n_\ell}||]/3, \text{ then}$$

$$\begin{aligned} 3\, ||T_1x_{n_\ell} - T_2x_{n_k}|| \le\ & ||x_{n_\ell} - T_1x_{n_\ell}|| + ||T_1x_{n_\ell} - T_2x_{n_k}|| \\ & + ||x_{n_k} - T_2x_{n_k}|| + ||T_2x_{n_k} - T_1x_{n_\ell}|| , \end{aligned}$$

which implies

(11) $\quad ||T_1x_{n_\ell} - T_2x_{n_k}|| \le ||x_{n_\ell} - T_1x_{n_\ell}|| + ||x_{n_k} - T_2x_{n_k}||.$

If

$$||T_1x_{n_\ell} - T_2x_{n_k}|| \le [||x_{n_\ell} - x_{n_k}||+||x_{n_\ell} - T_1x_{n_\ell}|| + ||x_{n_k} - T_2x_{n_k}||]/3 ,$$

it follows, in a similar manner, that (11) holds. Therefore, in all cases, (11) is satisfied.

Therefore,

$$||T_1x_{n_k} - T_1x_{n_\ell}|| \le ||T_1x_{n_k} - x_{n_k}|| + ||x_{n_k} - T_2x_{n_k}|| + ||x_{n_\ell} - T_1x_{n_\ell}|| + ||x_{n_k} - T_2x_{n_k}||,$$

which tends to 0 as $k \to \infty$. Therefore $\{T_1x_{n_k}\}$ is a Cauchy sequence and hence it converges, say, to u. Consequently

$$\lim x_{n_k} = \lim T_1x_{n_k} = u.$$

Also,

$$||u-T_2u|| \le ||u-x_{n_k}|| + ||x_{n_k}-T_1x_{n_k}|| + ||T_1x_{n_k}-T_2u|| \le ||u-x_{n_k}|| + ||x_{n_k}-T_1x_{n_k}||$$

$$+ \max\{(||x_{n_k} - T_1x_{n_k}|| + ||u - T_2u||)/2,$$

$$(||(x_{n_k} - T_2u|| + ||u - T_1x_{n_k}||)/3,$$

$$(||x_{n_k} - u|| + ||x_{n_k} - T_1x_{n_k}|| + ||u - T_2u||)/3\}.$$

Taking the limit as $k \to \infty$, we obtain $||u - T_2u|| = 0$. Therefore, $u = T_2u$.

Now,

$$||u - T_1u|| \le ||u - T_2u|| + ||T_2u - T_1u||$$

$$\le \max\{(||u - T_1u|| + ||u - T_2u||)/2,$$

$$(||u - T_2u|| + ||u - T_1u||)/3,$$

$$(||u - u|| + ||u - T_1u|| + ||u - T_2u||)/3\}$$

This implies $||u - T_1u|| = 0$. Therefore, $u = T_1u$.

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