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ON THE DIFFERENTIABILITY OF T(r,f)

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ABSTRACT. It is well known that $T(r,f)$ is differentiable at least for $r > r_0$. We show that, in fact, $T(r,f)$ is differentiable for all but at most one value of r, and if $T(r,f)$ fails to have a derivative for some value of r, then f is a constant times a quotient of finite Blaschke products.

KEY WORDS AND PHRASES. Nevanlinna Theory, Nevanlinna Characteristic Function.
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1. INTRODUCTION.

In this paper we will discuss the differentiability of the Nevanlinna characteristic function, $T(r,f)$, for f meromorphic in $|z| < R \leq \infty$. As early as 1929 Cartan stated in [1] that $T(r,f)$ is differentiable, and gave a formula for $dT(r,f)/d \log r$, but did not indicate whether the derivative may fail to exist for some values of r. We will show that $T(r,f)$ is differentiable for all but at most one value of r, and if the derivative fails to exist for some value of r, then f is a constant times a quotient of finite Blaschke products.

2. DEFINITIONS AND STATEMENT OF THE THEOREM.

We begin by defining the standard Nevanlinna functionals.

DEFINITION. Let f be a meromorphic function in $|z| < R \leq \infty$. Then for $r < R$,

1. $n(r,f) = n(r,\infty,f) =$ the number of poles of f in $|z| \leq r$, and
   $n(r,a,f) = n(r, \frac{1}{f-a}) =$ the number of solutions to the equation $f(z) = a$ in $|z| \leq r$.
2. $$N(r,f) = N(r,\infty,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}\, dt + n(0,f)\log r, \text{ and}$$
   $$N(r,a,f) = N(r,\frac{1}{f-a}).$$
3. $$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta \text{ , and}$$
   $$m(r,a,f) = m(r, \frac{1}{f-a}) .$$
4. $T(r,f) = m(r,f) + N(r,f)$, and
   $$T(r,a,f) = T(r, \frac{1}{f-a}) = m(r, \frac{1}{f-a}) + N(r, \frac{1}{f-a})$$

For the purposes of this paper we define the additional functional

5. $n^-(r,a,f) = n(r,a,f) - \{$the number of solutions to $f(z) = a$ on $|z| = r\}$.

We note that the derivative of $N(r,a,f)$ from the right with respect to log r is $n(r,a,f) - n(0,a,f)$, and the derivative from the left with respect to log r is $n^-(r,a,f) - n(0,a,f)$. Thus, $\frac{d\,N(r,a,f)}{d \log r} = n(r,a,f) - n(0,a,f)$ provided $f(z) \neq a$ on $|z| = r$. Since $N(r,a,f)$ is continuous and $n(r,a,f) - n(0,a,f)$ is monotonically increasing, $N(r,a,f)$ is an increasing convex function of log r.

The following lemma, which we state without proof, gives a characterization of $T(r,f)$ which has proved to be of great importance in the development of Nevanlinna theory. We will base our discussion of the differentiability of $T(r,f)$ largely on this lemma.

CARTAN'S LEMMA. If f is meromorphic in $|z| < R$ and $0 < r < R$, then

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta},f)\, d\theta + \log^+|f(0)|.$$

Since $N(r,a,f)$ is an increasing convex function of log r, it follows from Cartan's Lemma that $T(r,f)$ is also a convex, increasing function of log r. By a well known theorem concerning convex functions, $T(r,f)$ has derivatives from the right and from the left for all $r > 0$, and is differentiable for all but at most countably many values of r. We give an example to show that $T(r,f)$ need not be differentiable for all values of r.

EXAMPLE. Let $f(x) = 2\,\frac{1 + iz/2}{1 + 2iz}$. The function f is a one-to-one map of the extended plane, $\hat{C}$, onto $\hat{C}$, and takes the unit circle onto the unit circle. Thus, for all real θ,

$$n(r,e^{i\theta},f) = \begin{cases} 0 \text{ if } r < 1 \\ 1 \text{ if } r \geq 1 \end{cases}.$$

Also,

$$N(r,e^{i\theta},f) = \int_0^r \frac{n(t,e^{i\theta},f)}{t}\,dt = \begin{cases} 0 & \text{if } r < 1 \\ \log r & \text{if } r > 1 \end{cases}.$$

and by Cartan's Lemma

$$T(r,f) - \log 2 = \frac{1}{2\pi}\int_0^{2\pi} N(r,e^{i\theta},f)\,d\theta = \begin{cases} 0 & \text{if } r < 1 \\ \log r & \text{if } r \geq 1 \end{cases}.$$

Thus, $T(r,f)$ is not differentiable at $r = 1$.

That this example is representative of the class of functions for which $T(r,f)$ fails to have a derivative for some value of r, is evident from the proof of the following theorem.

THEOREM. Let f be a nonconstant meromorphic function in $|z| < R \leq \infty$. Then $T(r,f)$ is differentiable for all but at most one value of $r < R$. If $T(r,f)$ fails to have a derivative at $r = r_0$, then f is a constant times a quotient of finite Blaschke products.

3. PROOF OF THE THEOREM.

STEP 1. In this part of the proof we will use the fact that for $r < r_1 < R$ $n(r,a,f)$ is uniformly bounded for all $a \in C$ by a finite constant depending only on r_1.

Suppose that $0 < r' < r_0 < r'' < R$, and consider a sequence $\{r_k\}$ satisfying $r' < r_k < r''$, $r_k \neq r_0$ for all k and $\lim\limits_{k\to\infty} r_k = r_0$.

By Cartan's Lemma we have

$$\lim_{k\to\infty} \frac{T(r_k,f) - T(r_0,f)}{r_k - r_0} = \lim_{k\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{N(r_k,e^{i\theta},f) - N(r_0,e^{i\theta},f)}{r_k - r_0} \right] d\theta. \quad (3.1)$$

Since $n(r,a,f) < K(r'')$ for all $a \in C$ and all $r \leq r''$, the integrand of the integral in (3.1) is

$$\left(\int_0^{r_k} n(t,e^{i\theta},f)t^{-1}\, dt - \int_0^{r_0} n(t,e^{i\theta},f)t^{-1}\, dt\right)(r_k - r_0)^{-1}$$

$$= \frac{1}{r_k - r_0} \int_{r_0}^{r_k} n(t,e^{i\theta},f)t^{-1}\, dt < K(r'')\, \frac{\log r_k - \log r_0}{r_k - r_0} < K(r',r''),$$

where $K(r',r'')$ is a constant depending only on r' and r''. Therefore, by the Lebesgue Dominated Convergence Theorem

$$\lim_{k\to\infty} \frac{T(r_k,f) - T(r_0,f)}{r_k - r_0} = \frac{1}{2\pi} \int_0^{2\pi} \left[\lim_{k\to\infty} \frac{N(r_k,e^{i\theta},f) - N(r_0,e^{i\theta},f)}{r_k - r_0} \right] d\theta,$$

provided the limit in the integrand above exists off a set of Lebesgue measure zero. From the definition of $N(r,a,f)$ we have that

$$\lim_{k\to\infty} \frac{N(r_k,e^{i\theta},f) - N(r_0,e^{i\theta},f)}{r_k - r_0} = \frac{n(r_0,e^{i\theta},f)}{r_0},$$

provided $f(r_0e^{i\phi}) \neq e^{i\theta}$ for all $0 \leq \phi \leq 2\pi$. Therefore, if $|f(r_0e^{i\phi})| = 1$ only on a set of measure zero then, since $\{r_k\}$ is an arbitrary sequence converging to r_0, we have

$$r_0 \left. \frac{dT(r,f)}{dr} \right|_{r = r_0} = \frac{1}{2\pi} \int_0^{2\pi} n(r_0,e^{i\theta},f)\, d\theta.$$

STEP 2. Suppose that for some $r_0 < R$, $|f(r_0e^{i\phi_j})| = 1$ for $j = 1,2,3\ldots$ and $\lim_{j\to\infty} \phi_j = \phi_0$. Let L_1 be a linear fractional transformation mapping the real line to $|z| = r_0$ and L_2 a linear fractional transformation mapping $|z| = 1$ to the real line. Let $g = L_2 \circ f \circ L_1$. Then g is meromorphic in a neighborhood of the real axis and there exists a finite x_0 which is a limit point of $\{x : g(x) \text{ is real}\}$.

From elementary power series considerations it follows that g is (extended) real-valued everywhere on the real axis. Hence, $|f(r_0e^{i\phi})| = 1$ for all $0 \leq \phi \leq 2\pi$. Thus, for an interval of θ values in $[0,2\pi)$ having positive length

$$n(r_0,e^{i\theta},f) \geq n^-(r_0,e^{i\theta},f) + 1. \tag{3.2}$$

If $r_k \uparrow r_0$, then as in Step 1 above

$$\lim_{k\to\infty} \frac{T(r_k,f) - T(r_0,f)}{r_k - r_0} = \frac{1}{2\pi}\int_0^{2\pi} \left[\lim_{k\to\infty} \frac{N(r_k,e^{i\theta},f) - N(r_0,e^{i\theta},f)}{r_k - r_0} \right] d\theta \tag{3.3}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} n^-(r_0,e^{i\theta},f)\, r_0^{-1}\, d\theta$$

Similarly, if $r_k \downarrow r_0$, then

$$\lim_{k\to\infty} \frac{T(r_k,f) - T(r_0,f)}{r_k - r_0} = \frac{1}{2\pi}\int_0^{2\pi} n(r_0,e^{i\theta},f) r_0^{-1}\, d\theta\ . \tag{3.4}$$

By (3.2) the limits in (3.3) and (3.4) are not equal and hence $T(r,f)$ fails to have a derivative at $r = r_0$.

To summarize steps 1 and 2, either $|f(z)| = 1$ finitely often on $|z| = r_0$, in which case $T(r,f)$ possesses a derivative at $r = r_0$, or $|f(z)| = 1$ everywhere on $|z| = r_0$, in which case $T(r,f)$ fails to have a derivative at $r = r_0$.

STEP 3. Suppose $|f(z)| = 1$ everywhere on $|z| = r_0$. Let the zeros and poles of f, counting multiplicity, in $|z| < r_0$ be $\{a_1,a_2, \ldots, a_M\}$ and $\{b_1,b_2, \ldots, b_N\}$, respectively. Define

$$B(z) = \prod_{j=1}^{M} \left(\frac{|a_j|}{r_0} \cdot \frac{1 - za_j^{-1}}{1 - \bar{a}_j zr_0^{-2}}\right) \Big/ \prod_{j=1}^{N} \left(\frac{|b_j|}{r_0} \cdot \frac{1 - zb_j^{-1}}{1 - \bar{b}_j zr_0^{-2}}\right).$$

The function $P(z) = \frac{|a|}{r_0} \cdot \frac{1 - za^{-1}}{1 - \bar{a}zr_0^{-2}}$ is a Blaschke factor having a zero at $z = a$ and a pole at $z = r_0^2(\bar{a})^{-1}$, and having modulus one on $|z| = r_0$. Thus, $\frac{f(z)}{B(z)}$ and $\frac{B(z)}{f(z)}$ are holomorphic in $|z| \leq r_0$, have no zeros or poles in $|z| \leq r_0$, and have modulus one on $|z| = r_0$. It follows from the maximum modulus theorem that $f(z) = \alpha B(z)$, where $|\alpha| = 1$.

It remains only to show that $|f(z)| = 1$ on $|z| = r$ for at most one value of r. Note that from the above argument, if $|f(z)| = 1$ for $|z| = r_0$, then f(z) has as many zeros (poles) in $|z| < r_0$ as it has poles (zeros) in $|z| > r_0$. If $|f(z)| = 1$ for $|z| = r_0' > r_0$, then by the same argument f(z) has as many zeros (poles) in $|z| < r_0'$ as it has poles (zeros) in $|z| = r_0'$. It follows readily that f(z) must have no zeros or poles in $r_0 < |z| < r_0'$. Therefore, f(z) and $\frac{1}{f(z)}$ are analytic in $r_0 < |z| < r_0'$ and both have modulus one on $|z| = r_0$ and $|z| = r_0'$. By the Maximum Modulus Theorem, f(z) must be a constant, which contradicts one of the hypotheses. Hence, $|f(z)| = 1$ on $|z| = r$ for at most one value of r, which completes the proof of the theorem.

If we let $\nu(r,f)$ be the number of solutions of the equation $|f(z)| = 1$ for $|z| = r$, then we have shown that $\nu(r,f) < \infty$ for all but at most one value of r. Questions concerning the growth of $\nu(r,f)$ have been posed in [2] and [3], and these questions have been investigated by the author and J. Miles in [4] and [5].

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