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# BAZILEVIC FUNCTIONS OF TYPE $\beta$

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ABSTRACT. In this paper, a new coefficient result for the Bazilevič functions of type $\beta$ is obtained.

*KEY WORDS AND PHRASES. Bazilevič functions, Starlike, Coefficients, Close-to-convex, Univalent functions.*

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## 1. INTRODUCTION.

Let S denote the class of functions f which are analytic and univalent in $E = \{z: |z| < 1\}$ and which satisfy $f(0) = 0$ and $f'(0) = 1$. Let $S^*$ be the class consisting of starlike functions. Bazilevič [1] introduced a class of analytic functions f defined by the following relation. For $z \in E$, let

$$f(z) = \left\{ \frac{\beta}{1+a^2} \int_0^z (H(\xi) - ai)\, \xi^{\frac{-\beta ai}{1+a^2} - 1}\, g^{\beta/1+a^2}(\xi)\, d\xi \right\}^{\frac{1+ai}{\beta}}, \tag{1.1}$$

where a is real, $\beta > 0$, $\mathrm{Re}H(z) > 0$ and $g \in S^*$. Such functions, he showed, are univalent [1]. With $a = 0$ in (1.1), we have [2] for $z \in E$,

$$\mathrm{Re}\, \frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} > 0 . \tag{1.2}$$

We shall denote this class of functions by $B(\beta)$. We notice that, if $\beta = 1$ in (1.2), we have the class of close-to-convex functions first introduced by Kaplan [3].

## 2. MAIN RESULTS.

Denote $M(r,f) = \max_{|z=r|} |f(z)|$, $0 \le r < 1$ and $M(r,f) \le (1 - r)^{-\alpha}$, $0 \le \alpha \le 2$. Thomas [2] has proved that $n|a_n| \le K(\alpha,\beta)n^{\alpha}$. We improve his result as follows:

THEOREM 1. Let $f \in B(\beta)$, for $0 < \beta \le 1$, and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

Then, for $n \ge 2$,

$$n|a_n| \le A(\beta)M\left(\frac{2n-1}{2n}, f\right),$$

where $A(\beta)$ is a constant depending only upon $\beta$.

PROOF. From (1.2), we can write

$$zf'(z) = f^{1-\beta}(z)\, g^{\beta}(z)h(z), \ \operatorname{Re} h(z) > 0;\ g \in S^* . \tag{2.3}$$

Thus,

$$(zf'(z))' = (1 - \beta)\, f^{1-\beta}(z)f'(z)g^{\beta}(z)h(z) + \beta f^{1-\beta}(z)g^{\beta-1}(z)g'(z)h(z) + f^{1-\beta}(z)g^{\beta}(z)h'(z) . \tag{2.4}$$

Since, with $z = re^{i\theta}$, Cauchy's theorem gives

$$n^2 a_n = \frac{1}{2\pi r^n}\int_0^{2\pi} z(zf'(z))'\, e^{-in\theta}\, d\theta ,$$

we have from (2.4),

$$n^2|a_n| \le \frac{1}{2\pi r^n}\left\{(1 - \beta)\int_0^{2\pi} |zf'(z)\ f^{-\beta}(z)g^{\beta}(z)h(z)|d\theta + \beta\int_0^{2\pi} |zg'(z)f^{1-\beta}(z)g^{\beta-1}(z)h(z)|d\theta + \int_0^{2\pi} |f^{1-\beta}(z)g^{\beta}(z)zh'(z)\ d\theta\right\}$$

$$= \frac{1}{r^n}|I_1 + I_2 + I_3|, \text{ say.}$$

Now,

$$I_1 = \frac{(1 - \beta)}{2\pi}\int_0^{2\pi} |zf'(z)f^{-\beta}(z)g^{\beta}(z)h(z)|d\theta$$

$$= \frac{(1 - \beta)}{2\pi}\int_0^{2\pi} |f'(z)|^2|f(z)|^{-1}d\theta, \text{ using (2.3).}$$

In order to estimate this integral, we use the following well-known result [4, p. 46].

Suppose that f is a really mean p-valent in E and that $\frac{1}{2} \le r < 1$, $0 < \lambda \le 2$. Then there exists ρ such that $2r - 1 \le \rho \le r$ and

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 \, |f(\rho e^{i\theta})|^{\lambda-2} d\theta \le \frac{4PM(r,f)^\lambda}{\lambda(1-r)} . \tag{2.5}$$

With $\rho = 1$ and $\lambda = 1$ in (2.5), we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 \, |f(\rho e^{i\theta})|^{-1} d\theta \le \frac{4M(r,f)}{(1-r)} .$$

Since $r < \frac{1+\rho}{2}$ and $M(r,f)$ is an increasing function, $M(r,f) \le M(\frac{1+\rho}{2}, f)$ .

Also,

$$\frac{1}{1-r} \le \frac{2}{1-\rho} \quad \text{since } 2r - 1 \le \rho \le r .$$

Thus

$$I_1 \le 8(1-\beta) \frac{M(\frac{1+\rho}{2}, f)}{(1-\rho)} .$$

Choosing $\rho = 1 - \frac{1}{n}$, we obtain for $n \ge 2$, see [5, p. 238, 240],

$$I_1 \le 8(1-\beta) \, M\left(\frac{2n-1}{2n}, f\right). \; n .$$

For $z = re^{i\theta}$,

$$I_2 = \frac{\beta}{2\pi} \int_0^{2\pi} |zg'(z) f^{1-\beta}(z) g^{\beta-1}(z) h(z)| d\theta$$

$$= \frac{\beta r}{2\pi} \int_0^{2\pi} |f'(z)\phi(z)| d\theta, \text{ where } \phi(z)g(z) = zg'(z);\ \mathrm{Re}\,\phi(z) > 0.$$

Applying the Schwarz inequality, we have

$$I_2 \le \frac{\beta r}{2\pi} \left( \int_0^{2\pi} |f'(z)|^2 d\theta \right)^{\frac{1}{2}} \left( \int_0^{2\pi} |\phi(z)|^2 d\theta \right)^{\frac{1}{2}} .$$

Now

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(z)|^2 \, d\theta = \sum_{n=1}^{\infty} n^2 |a_n|^2 \, r^{2n-2} \le \sum_{n=1}^{\infty} n |a_n|^2 \, r^n , \max n \, r^{n-2} .$$

Since the function $\log (nr^n)$ has a maximum at a point $n_o = \frac{1}{\log \frac{1}{r}}$, we have

$$\log nr^n \le \log n_o r^{n_o} = \log \frac{1}{e \log \frac{1}{r}} ;$$

$$\text{i.e., } nr^{n-2} \le \frac{1}{er^2 \log \frac{1}{r}} \le \frac{1}{er^2(1-r)} . \tag{2.6}$$

Also, it is well-known [6, p. 42] that $M(r,f) \le \frac{4}{r} M(r^2,f)$ , (2.7)

and that the area principal for univalent functions gives

$$A(r,f) \le \pi M^2(r,f), \text{ (see [7, p. 215]).} \tag{2.8}$$

Using (2.6), (2.7) and (2.8), we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} |f'(z)|^2 d\theta \le \frac{16}{e} \frac{M(r,f)^2}{r^3(1-r)} . \tag{2.9}$$

Since Re $\phi(z) > 0$,

$$\phi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) = 1 + \sum_{n=1}^{\infty} c_n z^n \text{ with } |c_n| \le 2; \qquad \text{see [4].}$$

Thus

$$\frac{1}{2\pi} \int_0^{2\pi} |\phi(z)|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \le 1 + 4 \sum_{n=1}^{\infty} r^{2n} = \frac{1 + 3r^2}{1 - r^2} \tag{2.10}$$

From (2.9) and (2.10), we have

$$I_2 \le \frac{4\beta}{\sqrt{er}} \frac{M(r,f)}{(1-r)} \left[\frac{1 + 3r^2}{1 + r}\right]^{\frac{1}{2}} \le \frac{4\sqrt{2}\beta}{\sqrt{er}} \frac{M(r,f)}{(1-r)} .$$

Since $r \le \frac{1+r}{2}$ and $M(r,f)$ is an increasing function, we have for $r = 1 - \frac{1}{n}$ and $n \ge 2$

$$I_2 \le \frac{4\sqrt{2}\beta}{\sqrt{er}} M\left(\frac{2n-1}{2n}, f\right). n$$

$$\le \frac{8\beta}{\sqrt{e}} M\left(\frac{2n-1}{2n}, f\right). n$$

Finally, since $0 < \beta \le 1$, $z = re^{i\theta}$ and

$$\begin{aligned} I_3 &= \frac{1}{2\pi} \int_0^{2\pi} |f^{1-\beta}(z) g^{\beta}(z)\, zh'(z)| d\theta \\ &\le M^{1-\beta}(r,f) \cdot \frac{1}{2\pi} \int_0^{2\pi} |g^{\beta}(z)\, zh'(z)| d\theta \\ &\le M^{1-\beta}(r,f) \frac{2r}{1-r^2} \frac{1}{2\pi} \int_0^{2\pi} |g^{\beta}(z)| \mathrm{Re}\, h(z) d\theta . \end{aligned}$$

Since it is known [8] that $|zh'(z)| \le \frac{2r\mathrm{Re}h(z)}{1 - r^2}$ , then by (3)

$$I_3 \le M^{1-\beta}(r,f)\ \frac{2r}{1-r^2}\ \mathrm{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} f^{\beta-1}(z)f'(z)e^{-i\arg g^{\beta}(z)}d\theta\right\}$$

Integrating by parts and using the fact that g is starlike, we obtain

$$I_3 \le M^{1-\beta}(r,f).\ \beta\ \frac{M^{\beta}(r,f)}{1-r}$$

$$\le \beta\ \frac{M(\frac{1+r}{2},f)}{1-r}$$

$$= \beta\ M(\frac{2n-1}{2n},f).\ n, \text{ on choosing } r = 1-\frac{1}{n},\ n \ge 2\ .$$

Thus, for $n \ge 2$

$$n|a_n| \le e\left\{8((1-\beta)+\frac{8\beta}{\sqrt{e}}+\beta\right\} M(\frac{2n-1}{2n},f), \text{ (see [4, p. 45])}.$$

This completes the proof.

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