

IDENTITIES INVOLVING ITERATED INTEGRAL TRANSFORMS

CYRIL NASIM

Department of Mathematics and Statistics
The University of Calgary
Calgary, Alberta Canada T2N 1N4

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ABSTRACT. A number of identities involving iterated integral transforms are established, making use of the fact that a function which is a linear combination of the Macdonald's function $K_\nu(z)$, where z is a complex variable, is a Fourier kernel.

KEY WORDS AND PHRASES. Macdonald's function, Fourier kernel, Mellin transform, Hankel transform, Laplace transform.

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1. INTRODUCTION.

The object of this note is to establish various identities involving integral operators. The integral operators are the integral transforms with respect to the function $K_\nu(z)$, where $K_\nu(z)$ is the Macdonald's function of order ν and argument z , a complex variable. Some functional relations are deduced, as special cases, which show the inter-relations among more familiar Fourier Sine, Fourier Cosine, and Laplace transforms.

2. THE KERNEL.

Let $y = x^{1/2} K_\nu(\theta x)$, with θ a constant and $|\nu| < 1$.

Then $y'' - \frac{\nu^2 - 1/4}{x^2} y = \theta^2 y$

or $\left(D^2 - \frac{\nu^2 - 1/4}{x^2} \right) y = \theta^2 y$, $D \equiv \frac{d}{dx}$.

Whence $\left(D^2 - \frac{\nu^2 - 1/4}{x^2} \right)^k y = \theta^{2k} y$, $k = 0, 1, 2, \dots$.

Now, if we set $\theta = i e^{i\pi m/k}$, $0 \leq m \leq 2k-1$,

then $y = x^{\frac{1}{2}} K_{\nu}(ie^{i\pi m/k} x)$,

satisfies a k -fold Bessel equation:

$$\left(D^2 - \frac{\nu^2 - 1/4}{x^2}\right)^k y = (-1)^k y \quad (2.1)$$

It is not difficult to see that if x is a complex variable, then every point of (2.1) is regular except for a singularity at $x = 0$. Now consider a function of the form

$$G_k(x) = \sum_{m=0}^{2k-1} B_m x^{\frac{1}{2}} K_{\nu}(ie^{i\pi m/k} x), \quad |\nu| < 1.$$

The functions are an extension of the functions which were first noted by Guinand.

As a special case when $k = 2$, chose the coefficients as

$$B_0 = B_2 = -\frac{1}{\pi}, \quad B_1 = 0 \text{ and } B_3 = -\frac{2}{\pi} \cos \frac{1}{2} \nu \pi.$$

Then we obtain

$$\begin{aligned} G_2(x) &= -\frac{1}{\pi} x^{\frac{1}{2}} \{K_{\nu}(ix) + K_{\nu}(-ix) + 2\cos(\frac{1}{2}\nu\pi)K_{\nu}(x)\} \\ &= K_{\nu}(x), \text{ say,} \end{aligned} \quad (2.2)$$

and we have

THEOREM 2.1. $y = k_{\nu}(x)$ is a solution of

$$\left(D^2 - \frac{\nu^2 - 1/4}{x^2}\right)^2 y = y, \quad 0 < x < \infty,$$

the two-fold Bessel equation.

The function $k_{\nu}(x)$ is of special interest to us here and we shall develop its properties further.

Using the representations [3]

$$K_{\nu}(x) = \frac{\pi}{2} \operatorname{cosec} \nu\pi \{I_{-\nu}(x) - I_{\nu}(x)\}$$

and

$$Y_{\nu}(x) = \operatorname{cosec} \nu\pi \{\cos \nu\pi J_{\nu}(x) - J_{-\nu}(x)\},$$

where J_{ν} , I_{ν} and Y_{ν} are the usual Bessel functions, equation (2.2), can be written as

$$k_{\nu}(x) = x^{\frac{1}{2}} \left\{ \sin \frac{1}{2}\nu\pi J_{\nu}(x) + \cos \frac{1}{2}\nu\pi (Y_{\nu}(x) + \frac{2}{\pi} K_{\nu}(x)) \right\}.$$

These functions arise as kernels in divisor summation formulae of the Hardy-Landau

type, involving number theoretic function $\sigma_k(n)$, the number of k th powers of the divisor of n , [4]. If we put $\nu = \pm \frac{1}{2}$, we have

$$k_{\pm\frac{1}{2}}(x) = \pi^{-\frac{1}{2}}(\cos x - \sin x + e^{-x}) ,$$

which obviously satisfies the differential equation

$$D^4 y = y .$$

Next, the Mellin transform of $x^{\frac{1}{2}}K_{\nu}(\alpha x)$ is given by

$$m\{x^{\frac{1}{2}}K_{\nu}(\alpha x)\} = \alpha^{-s-\frac{1}{2}} 2^{s-3/2} \Gamma(\frac{1}{2}s + \frac{1}{2}\nu + \frac{1}{4}) \Gamma(\frac{1}{2}s - \frac{1}{2}\nu + \frac{1}{4}) ,$$

where $Re\ s > |Re\ \nu| - \frac{1}{2}$, [5], whence the Mellin transform of $k_{\nu}(x)$ defined in (2.2) is given by

$$k_{\nu}^*(s) = -\frac{2^{s-3/2}}{\pi} \Gamma(\frac{1}{2}s + \frac{1}{2}\nu + \frac{1}{4}) \Gamma(\frac{1}{2}s - \frac{1}{2}\nu + \frac{1}{4}) \\ (i^{-s-\frac{1}{2}} + (-i)^{-s-\frac{1}{2}} + 2\cos \frac{1}{2}\nu\pi) .$$

On simplifying, we have

$$k_{\nu}^*(s) = -\frac{2^{s+\frac{1}{2}}}{\pi} \Gamma(\frac{1}{2}s + \frac{1}{2}\nu + \frac{1}{4}) \Gamma(\frac{1}{2}s - \frac{1}{2}\nu + \frac{1}{4}) \\ \cos \frac{1}{4}\pi(s + \nu + \frac{1}{2}) \cdot \cos \frac{1}{4}\pi(s - \nu + \frac{1}{2}) ,$$

for $Re\ s > |Re\ \nu| - \frac{1}{2}$.

By repeated use of the relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} ,$$

it is not a difficult matter to see that

$$k_{\nu}^*(s)k_{\nu}^*(1-s) = 1 .$$

Hence,

THEOREM 2.2. The function $k_{\nu}^*(x)$ defined by equation (2.2) is a Fourier kernel, [6].

If we define the transformation $T[f]$ by

$$T[f] = \int_0^\infty k_\nu(xt)f(t)dt, \quad |\nu| < 1, \tag{2.3}$$

then $T^2[f] = f$ and T is involutory since $T^2 = I$, the identity transformation. Making use of the asymptotic expansions of the Macdonald's function $K_\nu(z)$, we have

$$k_\nu(x) = O(e^{-x}), \quad x \rightarrow \infty$$

and
$$k_\nu(x) = O(x^{-\nu+1/2}), \quad x \rightarrow 0.$$

Therefore
$$k_\nu(x) \in L^2(0, \infty), \quad \text{for } |\nu| < 1.$$

Now, if we take $f(x) \in L^2(0, \infty)$, the integral defining the transformation T exists and is in fact absolute convergent. Thus T is a bounded transformation on L^2 -space for $|\nu| < 1$.

3. THE OPERATOR.

We shall now define the transformation T in operator notation. Denote the operators K_ν and $K_{\nu,i}$ respectively by

$$K_\nu[f] \equiv K_\nu\{f(x); x\} = \int_0^\infty \sqrt{xt} K_\nu(xt)f(t)dt$$

and
$$K_{\nu,i}[f] \equiv K_{\nu,i}\{f(x); ix\} = \int_0^\infty \sqrt{xt} K_\nu(ixt)f(t)dt,$$

where $f \in L^2(0, \infty)$ and $|\nu| < 1$, with $K_\nu(z)$ being the Macdonald's function. Then the transformation T can be expressed, in operator form, as

$$\begin{aligned} T[f] &= \int_0^\infty k_\nu(xt)f(t)dt \\ &= -\frac{1}{\pi} \left\{ \int_0^\infty \sqrt{xt} K_\nu(ixt)f(t)dt + \int_0^\infty \sqrt{xt} K_\nu(-ixt)f(t)dt \right. \\ &\quad \left. + 2\cos \frac{1}{2} \nu\pi \int_0^\infty \sqrt{xt} K_\nu(xt)f(t)dt \right\} \\ &= -\frac{1}{\pi}(K_{\nu,i} + K_{\nu,-i} + 2\cos \frac{1}{2} \nu\pi K_\nu)[f]. \end{aligned}$$

and we can write, symbolically,

$$T \equiv -\frac{1}{\pi}(K_{\nu,i} + K_{\nu,-i} + 2\cos \frac{1}{2} \nu\pi K_\nu).$$

Since $T^2 = I$, the identity transformation, we have

$$\frac{1}{\pi^2} (K_{\nu, i} + K_{\nu, -i} + 2\cos \frac{1}{2} \nu\pi K_{\nu})^2 = I$$

or

$$I = \frac{1}{\pi^2} \left\{ K_{\nu, i}^2 + K_{\nu, -i}^2 + 4\cos^2 \frac{1}{2} \nu\pi K_{\nu}^2 + K_{\nu, i}K_{\nu, -i} + K_{\nu, -i}K_{\nu, i} + 2\cos \frac{1}{2} \nu\pi (K_{\nu, i}K_{\nu} + K_{\nu, -i}K_{\nu} + K_{\nu}K_{\nu, i} + K_{\nu}K_{\nu, -i}) \right\} \tag{3.1}$$

The right-hand side is the linear combination of iterated transformations, which are bounded on L^2 -space for $|\nu| < 1$.

Now, using the standard result [5],

$$\int_0^{\infty} tK_{\nu}(\alpha t)K_{\nu}(\beta t)dt = \frac{\pi(\alpha\beta)^{-\nu}}{\sin \nu\pi} \left(\frac{\alpha^2\nu - \beta^2\nu}{\alpha^2 - \beta^2} \right),$$

where $|\nu| < 1$ and $Re(\alpha+\beta) > 0$, we have for example, if $f \in L^2(0, \infty)$,

$$\begin{aligned} K_{\nu, i}K_{\nu}[f] &= \frac{1}{\pi^2} \int_0^{\infty} \sqrt{xt} K_{\nu}(ixt)dt \int_0^{\infty} (ut)^{\frac{1}{2}} K_{\nu}(ut)f(u)du \\ &= \frac{1}{\pi^2} \int_0^{\infty} (xu)^{\frac{1}{2}} f(u)du \int_0^{\infty} tK_{\nu}(ixt)K_{\nu}(ut)dt \\ &= \frac{-(i)^{-\nu}}{\pi \sin \nu\pi} \int_0^{\infty} (xu)^{\frac{1}{2}-\nu} f(u) \frac{(ix)^{2\nu} - u^{2\nu}}{x^2 + u^2} du \\ &= -K_{\nu}K_{\nu, -i}[f]. \end{aligned}$$

The change of order of integration can be justified by absolute convergence. Thus, we obtain our first identity

$$K_{\nu, i}K_{\nu} + K_{\nu}K_{\nu, -i} = 0 \tag{3.2}$$

The identity given by (3.2) can alternatively be established by making use of the Mellin transform theory. That is, the Mellin transform of the iterated operator

$K_{\nu, i}K_{\nu}[f]$, is given formally by

$$\begin{aligned} m\{K_{\nu, i}K_{\nu}[f]\} &= m\left\{-\frac{1}{\pi} x^{\frac{1}{2}} K_{\nu}(ix); s\right\} m\left\{-\frac{1}{\pi} x^{\frac{1}{2}} K_{\nu}(x); 1-s\right\} f^*(s) \\ &= \frac{i^{-s-\frac{1}{2}}}{4\sin \frac{1}{2}\pi(s-\nu+\frac{1}{2})\sin \frac{1}{2}\pi(s+\nu+\frac{1}{2})} f^*(s), \end{aligned}$$

where $f^*(s)$ denotes the Mellin transform of $f(x)$. Also,

$$\begin{aligned} m\{K_{\nu}K_{\nu, -i}[f]\} &= m\left\{-\frac{1}{\pi} x^{\frac{1}{2}} K_{\nu}(x); s\right\} m\left\{-\frac{1}{\pi} x^{\frac{1}{2}} K_{\nu}(-ix); 1-s\right\} f^*(s) \\ &= \frac{(-i)^{s-3/2}}{4\sin \frac{1}{2}\pi(s-\nu+\frac{1}{2})\sin \frac{1}{2}\pi(s+\nu+\frac{1}{2})} f^*(s). \end{aligned}$$

Then
$$m\{(K_{\nu,i}K_{\nu} + K_{\nu}K_{\nu,-i})[f]\} = 0,$$

implying that

$$K_{\nu,i}K_{\nu} + K_{\nu}K_{\nu,-i} = 0,$$

as shown above. Similarly, one can show that

$$K_{\nu}K_{\nu,i} + K_{\nu,-i}K_{\nu} = 0, \quad (3.3)$$

a sort of conjugate of the identity in (3.2). Consider the representation

$$K_{\nu}(ix) = \frac{\pi}{2 \sin \nu\pi} (e^{-i\frac{1}{2}\nu\pi} J_{-\nu}(x) - e^{i\frac{1}{2}\nu\pi} J_{\nu}(x));$$

then

$$\begin{aligned} K_{\nu,i}[f] &= \int_0^{\infty} (xt)^{\frac{1}{2}} K_{\nu}(ixt) f(t) dt, \quad |\nu| < 1 \\ &= \frac{\pi}{2 \sin \nu\pi} \left\{ e^{-i\frac{1}{2}\nu\pi} \int_0^{\infty} (xt)^{\frac{1}{2}} J_{-\nu}(xt) f(t) dt \right. \\ &\quad \left. - e^{i\frac{1}{2}\nu\pi} \int_0^{\infty} (xt)^{\frac{1}{2}} J_{\nu}(xt) f(t) dt \right\} \\ &= \frac{\pi}{2 \sin \nu\pi} \left\{ e^{-i\frac{1}{2}\nu\pi} H_{-\nu}[f] - e^{i\frac{1}{2}\nu\pi} H_{\nu}[f] \right\}, \end{aligned}$$

where H_{ν} denotes the Hankel transform operator of order ν . Thus, in operator form,

$$K_{\nu,i} = \frac{\pi}{2 \sin \nu\pi} e^{-i\frac{1}{2}\nu\pi} H_{-\nu} - \frac{\pi}{2 \sin \nu\pi} e^{i\frac{1}{2}\nu\pi} H_{\nu} \quad (3.4)$$

and similarly

$$K_{\nu,-i} = \frac{\pi}{2 \sin \nu\pi} e^{i\frac{1}{2}\nu\pi} H_{-\nu} - \frac{\pi}{2 \sin \nu\pi} e^{-i\frac{1}{2}\nu\pi} H_{\nu}.$$

Now, after substituting for $K_{\nu,i}$ and $K_{\nu,-i}$ in (3.2) and rearranging, we have

$$e^{-i\frac{1}{2}\nu\pi} (H_{-\nu}K_{\nu} - K_{\nu}H_{\nu}) = e^{i\frac{1}{2}\nu\pi} (H_{\nu}K_{\nu} - K_{\nu}H_{-\nu}).$$

By comparing the real and imaginary parts and solving, we obtain the identities

$$K_{\nu}H_{\nu} = H_{-\nu}K_{\nu} \quad (3.5)$$

and

$$H_{\nu}K_{\nu} = K_{\nu}H_{-\nu} \quad (3.6)$$

Now,

$$\begin{aligned} H_{\frac{1}{2}}[f] &= \int_0^\infty (xt)^{\frac{1}{2}} J_{\frac{1}{2}}(xt) f(t) dt \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \sin(xt) f(t) dt \\ &= S[f], \end{aligned}$$

and similarly,

$$H_{-\frac{1}{2}}[f] = C[f],$$

where S and C are the usual Fourier sine and cosine transform respectively.

Also,

$$\begin{aligned} K_{\frac{1}{2}}[f] &= \int_0^\infty (xt)^{\frac{1}{2}} K_{\frac{1}{2}}(xt) f(t) dt \\ &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-xt} f(t) dt \\ &= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} L[f], \end{aligned}$$

where $L[f]$ is the Laplace transform.

Hence, setting $\nu = \pm\frac{1}{2}$ in (3.4), we obtain the relations,

$$LS = CL$$

and

$$LC = SL. \tag{3.7}$$

Incidentally, a general relation involving the operators K_ν and H_μ can be established:

$$H_\mu K_\nu = \operatorname{cosec} \nu\pi \left\{ \sin \frac{1}{2}(\nu-\mu)\pi K_\mu H_\nu + \sin \frac{1}{2}(\nu+\mu)\pi K_\mu H_{-\nu} \right\} \tag{3.8}$$

On setting $\mu = \pm\nu$, (3.8) yields the identities (3.5) and (3.6). Next, from the representation (3.4), we have symbolically

$$\begin{aligned} K_{\nu, i}^2 &= \left(\frac{\pi}{2\sin \nu\pi}\right)^2 \left(e^{-i\frac{1}{2}\nu\pi} H_{-\nu} - e^{i\frac{1}{2}\nu\pi} H_\nu \right)^2 \\ &= \left(\frac{\pi}{2\sin \nu\pi}\right)^2 (2\cos \nu\pi I - H_{-\nu} H_\nu - H_\nu H_{-\nu}) \end{aligned} \tag{3.9}$$

since $H_{-\nu}^2 = H_\nu^2 = I$, the identity operator, with H_ν being the Hankel transform.

Similarly, one can show that

$$K_{\nu, i}^2 = K_{\nu, -i}^2. \tag{3.10}$$

And, in the same vein, we have

$$K_{\nu, i} K_{\nu, -i} + K_{\nu, -i} K_{\nu, i} = 2 \left\{ \frac{\pi}{2 \sin \nu \pi} \right\}^2 \left\{ 2I - \cos \nu \pi (H_{\nu} H_{-\nu} + H_{-\nu} H_{\nu}) \right\} \quad (3.11)$$

Now, going back to equation (3.1) and using the results given by (3.2), (3.3), and (3.9) in (3.11) and simplifying, we have finally, for $|v| < 1$,

$$H_{\nu} H_{-\nu} + H_{-\nu} H_{\nu} - \left(\frac{2}{\pi} \sin \nu \pi \right)^2 K_{\nu}^2 = 2 \cos \nu \pi I. \quad (3.12)$$

An interesting relationship can be established by putting $\nu = \pm \frac{1}{2}$ in (3.12). Then

$$H_{\frac{1}{2}} H_{-\frac{1}{2}} + H_{-\frac{1}{2}} H_{\frac{1}{2}} - \frac{4}{\pi^2} K_{\frac{1}{2}}^2 = 0$$

or

$$SC + CS = \frac{2}{\pi} L^2, \quad (3.13)$$

where S, C , and L denote the Fourier sine, Fourier cosine and Laplace transforms respectively.

From (3.9) and (3.12), one can establish the identity

$$K_{\nu}^2 = -K_{\nu, i}^2 = -K_{\nu, -i}^2, \quad (3.14)$$

which, on setting $\nu = \pm \frac{1}{2}$, yields (3.13).

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