

## RESEARCH NOTES

### DUAL CHARACTERIZATION OF THE DIEUDONNÉ-SCHWARTZ THEOREM ON BOUNDED SETS

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(Received August 3, 1982)

**ABSTRACT.** The Dieudonné-Schwartz Theorem on bounded sets in a strict inductive limit is investigated for non-strict inductive limits. Its validity is shown to be closely connected with the problem of whether the projective limit of the strong duals is a strong dual itself. A counter-example is given to show that the Dieudonné-Schwartz Theorem is not in general valid for an inductive limit of a sequence of reflexive, Fréchet spaces.

**KEY WORDS AND PHRASES.** *Locally convex space, inductive and projective limit, barrelled space, bounded set.*

**1980 MATHEMATICS SUBJECT CLASSIFICATION CODE.** *Primary 46A12, Secondary 46A07.*

#### 1. INTRODUCTION

This paper is written for those with at least an elementary knowledge of the theory of locally convex spaces. A good reference is the book of Schaeffer [1]. Let  $E_1 \subset E_2 \subset \dots$  be a sequence of locally convex, Hausdorff, linear topological spaces such that each  $E_n$  is continuously contained in  $E_{n+1}$ , and such that the union  $E = \bigcup_{m=1}^{\infty} E_m$  is Hausdorff as a locally convex inductive limit. It is obvious that any bounded subset of a space  $E_n$  is also bounded in  $E$ . If each bounded subset of  $E$  arises in this way, we shall say that the DSP (Dieudonné-Schwartz Property) holds. A well-known theorem of Dieudonné-Schwartz states that the DSP holds provided each  $E_n$  is closed in  $E_{n+1}$  and has the topology inherited from  $E_{n+1}$  (see [1] or [4] II.6.5).

In duality theory an increasing sequence  $E_1 \subset E_2 \subset \dots$  corresponds to a decreasing sequence  $F_1 \supset F_2 \supset \dots$  where each  $F_n$  is dual to  $E_n$ . The intersection  $F = \bigcap_{m=1}^{\infty} F_m$ , endowed with the projective limit topology induced from the weak topologies  $\sigma(F_n, E_n)$ , may be identified with the dual of  $E$  relative to the weak topology  $\sigma(F, E)$  ([1] IV.4.5). In applications the strong topologies  $\beta(F_n, E_n)$  are often of interest, along with the projective limit  $\pi(F)$  induced by these on  $F$ . The strong topology  $\beta(F, E)$  is always at least as fine as  $\pi(F)$ . The problem of determining when  $\pi(F)$  equals  $\beta(F, E)$  turns out to be closely connected with determination of the validity of the DSP. A precise statement of this is given in the theorem below.

2. THE QIU PROPERTY

Recent work by Qiu [2] suggests a slight relaxation of the DSP. Let  $\mathcal{B}$  be the set of all subsets  $B$  of  $E$  such that  $B$  is bounded in some  $E_n$ . We say that the QP (Qiu Property) holds if each bounded subset of  $E$  is contained in the closure of some  $B \in \mathcal{B}$ .

For spaces  $V$  and  $W$  dual to one another, and a subset  $S$  of  $V$ , we write  $\overline{S}^V$  for the  $\sigma(V, W)$ -closure of  $S$  in  $V$  and  $S^{\circ W}$  for the polar of  $S$  in  $W$ . Thus, if  $\langle S \rangle$  denotes the convex hull of  $S$ , the Bipolar Theorem ([1] IV.1.5) states that  $\overline{\langle S \rangle}^V = (S^{\circ W})^{\circ V}$ . We note for use below that the polars of the closed, radial, convex bounded subsets of  $V$  are just the barrels of  $W$ , and vice versa.

**THEOREM.** A necessary and sufficient condition for the QP to hold is that  $\pi(F) = \beta(F, E)$ .

**PROOF.** Suppose first that  $\pi(F) = \beta(F, E)$ , and let  $B$  be an arbitrary bounded subset of  $E$ . Then  $B^{\circ F}$  is a barrel and so contains a  $\beta(F, E)$ -open neighborhood of  $0$ . From  $\pi(F) = \beta(F, E)$  now follows that there is a barrel  $A$  in some  $F_n$  such that  $A \cap F \subset B^{\circ F}$ . Letting  $D = A^{\circ E_n}$ , we see that  $D$  is bounded in  $E_n$  and  $D^{\circ F_n} = A$ . Because  $D^{\circ F}$  is just  $D^{\circ F_n} \cap F = A \cap F$ , we have  $D^{\circ F} \subset B^{\circ F}$ . Consequently,  $B \subset \overline{\langle D^{\circ F} \rangle}^E = (B^{\circ F})^{\circ E} \subset (D^{\circ F})^{\circ E}$ . Since the Bipolar Theorem guarantees that  $(D^{\circ F})^{\circ E}$  is just the closure of  $D$  in  $E$ , we have shown that the QP holds.

Now suppose that the QP holds, and let  $A$  be an arbitrary barrel of  $F$ . Then  $A^{\circ E}$  is bounded and so there exists a bounded set  $B$  of some  $E_n$  such that  $A^{\circ E}$  is in the closure  $\overline{B}^E$  of  $B$  in  $E$ . Since  $B^{\circ F} = B^{\circ F_n} \cap F$  and  $B^{\circ F_n}$  is a barrel (being the polar of a bounded set), it follows that  $B^{\circ F}$  is a  $\pi(F)$ -neighborhood of  $0$ . But we have

$A \circ E \subset \overline{B^E} \subset \overline{\langle B \rangle^E}$  so

$$B \circ F = (\overline{\langle B \rangle^E}) \circ F \subset (A \circ E) \circ F = A.$$

We have shown that  $\beta(F, E) \subset \pi(F)$ . The reverse inequality is evident.

Q.E.D.

### 3. COUNTER-EXAMPLE

It was demonstrated in [3] that the DSP holds when all the  $E_n$  are reflexive Banach spaces. The following example shows that, for reflexive Fréchet spaces, even the QP may fail to hold.

For each  $n \in \mathbf{N}$ , let  $D_n$  be the region  $\mathbb{R} \setminus \{1, 2, \dots, n\}$  and let  $E_n$  be the linear space of functions infinitely differentiable on  $D_n$ . For  $n, m \in \mathbf{N}$  let  $K_{n,m}$  be the compact set  $\{x \in D_n : |x| \leq m, |x - j| \geq \frac{1}{m} \text{ for all } j = 1, 2, \dots, n\}$  and, for each  $f \in E_n$ , let

$$\|f\|_{n,m} = \sup\{m |f^{(i)}(x)| : x \in K_{n,m}, i = 0, 1, \dots, m\}.$$

Then each  $E_n$ , equipped with the locally convex topology generated by the family  $\{\| \cdot \|_{n,m} : m = 0, 1, \dots\}$ , is a nuclear Fréchet space ([4] III.8.3). Hence each  $E_n$  is a Montel space ([4] III.7.2, Corollary 2) and thus reflexive. We proceed to show that  $E = \bigcup_{m=1}^{\infty} E_n$  does not have the QP.

For each  $n \in \mathbf{N}$ , and  $x \in \mathbb{R}$ , let  $f_n(x) = (x - n)^{n - \frac{1}{2}} e^{-(x-n)^2}$  and let  $c_n = \sup\{|f_n^{(i)}(x)| : x \in D_n \setminus [n - 1, n + 1], i = 0, 1, \dots, n - 1\}$ . Clearly, each  $f_n$  is in  $E_n$ . Let  $V$  be any neighborhood of 0 in  $E$ . Then, for some  $m \in \mathbf{N}$ , the  $\| \cdot \|_{1,m}$ -unit ball  $W$  of  $E_1$  is contained by  $V$ . Evidently  $\frac{1}{nc_n} f_n$  is in  $W$  for  $n = m + 1, m + 2, \dots$ . Consequently there exists  $k > 0$  such that  $h_n = \frac{1}{nc_n} f_n \in kV$  for all  $n \in \mathbf{N}$ —that is, then set  $B = \{h_n : n \in \mathbf{N}\}$  is bounded in  $E$ .

Let  $D$  be a bounded subset of one of the spaces  $E_n$ . Then the number

$$M = \sup\{|h^{(n+1)}(x)| : x \in [n + \frac{1}{2}, n + \frac{3}{2}], h \in D\} \quad (3.1)$$

is finite. Let  $p$  be the polynomial (with non-vanishing constant term) such that

$$h_{n+1}^{(n+1)}(x) = (x - n - 1)^{-\frac{1}{2}} p(x) e^{-(x-n-1)^2} \text{ for all } x \in D_{n+1}.$$

Evidently there exists some  $r > 2$  such that

$$\inf\{|h_{n+1}^{(n+1)}(x)| : x \in [n + 1 - \frac{1}{r}, n + 1 - \frac{1}{2r}]\} \geq M + 2. \quad (3.2)$$

Let  $K$  be any integer larger than  $\frac{1}{2r}$  and  $n + 1$ . For each  $m \in \mathbb{N}$ , let  $S_m$  be the  $\| \cdot \|_{m,k}$ -unit ball of  $E_n$ . Then, for each  $g \in S_m$ , we have

$$\sup\{|g^{(n+1)}(x)| : x \in [n + 1 - \frac{1}{k}, n + 1 - \frac{1}{2k}]\} \leq 1. \quad (3.3)$$

Evidently this last inequality also holds for all  $g$  in the convex hull  $H$  of the union  $\bigcup_{m=1}^{\infty} S_m$ . From (3.2) and (3.3) follows that the set  $h_{n+1} + H$  is a neighborhood of  $h_{n+1}$  in

$E$  such that, for each  $g \in h_{n+1} + H$ ,

$$\inf\{|g^{(n+1)}(x)| : x \in [n + 1 - \frac{1}{k}, n + 1 - \frac{1}{2k}]\} \geq M + 1. \quad (3.4)$$

Hence, (3.1) and (3.4) imply that  $D \cap (h_{n+1} + H) = \emptyset$ . Thus the QP does not hold.

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