

CONVOLUTIONS OF PRESTARLIKE FUNCTIONS

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ABSTRACT. The convolution of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. For $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z/(1-z)^{2(1-\gamma)}$, the extremal function for the class of functions starlike of order γ , we investigate functions h , where $h(z) = (f * g)(z)$, which satisfy the inequality $|(zh'/h) - 1| / |(zh'/h) + (1-2\alpha)| < \beta$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, for all z in the unit disk. Such functions f are said to be γ -prestarlike of order α and type β . We characterize this family in terms of its coefficients, and then determine extreme points, distortion theorems, and radii of univalence, starlikeness, and convexity. All results are sharp.

KEY WORDS AND PHRASES: Convolution, Starlike Functions, and Univalent Functions.

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1. INTRODUCTION.

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$. A function $f \in S$ is said to be starlike of order α and type β if the inequality

$$|(zf'/f) - 1| / |(zf'/f) + (1-2\alpha)| < \beta$$

holds for some $\alpha, \beta (0 \leq \alpha < 1, 0 < \beta \leq 1)$ and for all z in E . The class of all such functions shall be denoted by $S^*(\alpha, \beta)$. Note that $S^*(\alpha, 1) \equiv S^*(\alpha)$, the class of functions starlike of order α , and that $S^*(0, \beta)$ is a subclass of starlike functions studied by Padmanabhan [1]. For $f \in S^*(\alpha, \beta)$, $0 < \beta < 1$, the values of zf'/f lie in a disk centered at $(1 + (1-2\alpha)\beta^2)/(1-\beta^2)$ whose radius is $2\beta(1-\alpha)/(1-\beta^2)$.

The convolution or Hadamard product of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. A function f , analytic in E and normalized by $f(0) = f'(0) - 1 = 0$, is said to be in the class of prestarlike functions introduced by Ruscheweyh [2] if $f * s_{\gamma} \in S^*(\gamma)$, where $s_{\gamma}(z) = z/(1-z)^{2(1-\gamma)}$ with $0 \leq \gamma < 1$ is the well-known extremal function for the class $S^*(\gamma)$. We say that a normalized analytic function f is γ -prestarlike of order α and type β ($0 \leq \alpha < 1, 0 < \beta \leq 1$), denoted $R_{\gamma}(\alpha, \beta)$, if $f * s_{\gamma} \in S^*(\alpha, \beta)$.

Our main interest will be with functions f in $S^*(\alpha)$, $S^*(\alpha, \beta)$, or $R_{\gamma}(\alpha, \beta)$ that may be expressed as $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. We denote these classes, respectively, by $S^*[\alpha]$, $S^*[\alpha, \beta]$, and $R_{\gamma}[\alpha, \beta]$. The class $R_{\alpha}[\alpha, 1] \equiv R[\alpha]$ was studied in [3] while the class $S^*[\alpha, \beta]$ was investigated in [4]. For $\gamma = 1/2$ and $\beta = 1$, the class reduces to the family $S^*[\alpha]$ studied in [5].

We begin with a characterization of the class $R_{\gamma}[\alpha, \beta]$, from which we determine the extreme points, distortion properties, and radii of univalence, starlikeness, and convexity.

2. COEFFICIENT INEQUALITIES.

In the sequel, we set

$$C(\gamma, n) = \prod_{k=2}^n (k-2\gamma)/(n-1)! \quad (n = 2, 3, \dots), \quad (2.1)$$

so that s_{γ} may be written in the form $s_{\gamma}(z) = z/(1-z)^{2(1-\gamma)} = z + \sum_{n=2}^{\infty} C(\gamma, n) z^n$.

Note that $C(\gamma, n)$ is a decreasing function of γ , $0 \leq \gamma < 1$, with

$$\lim_{n \rightarrow \infty} C(\gamma, n) = \begin{cases} \infty, & \gamma < 1/2 \\ 1, & \gamma = 1/2 \\ 0, & \gamma > 1/2 \end{cases}$$

THEOREM 1. A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n > 0$, is in the class $R_{\gamma}[\alpha, \beta]$ if and only if

$$\sum_{n=2}^{\infty} \frac{[(n-1) + \beta(n+1-2\alpha)] C(\gamma, n) a_n}{2\beta(1-\alpha)} \leq 1. \tag{2.2}$$

PROOF. If $f \in R_{\gamma}[\alpha, \beta]$, then $g(z) = (f * s_{\gamma})(z) = z - \sum_{n=2}^{\infty} C(\gamma, n) a_n z^n \in S^*[\alpha, \beta]$, so that

$$\frac{|(zg'/g) - 1|}{|(zg'/g) + (1-2\alpha)|} = \left| \frac{\sum_{n=2}^{\infty} (n-1) C(\gamma, n) a_n z^{n-1}}{2(1-\alpha) - \sum_{n=2}^{\infty} (n+1-2\alpha) C(\gamma, n) a_n z^{n-1}} \right| < \beta \tag{2.3}$$

for all $z \in E$. Since the denominator in (2.3) is positive for small positive values of z and, consequently, for all $z, 0 < z < 1$, we let $z \rightarrow 1^-$ to obtain

$$\sum_{n=2}^{\infty} (n-1) C(\gamma, n) a_n \leq \beta [2(1-\alpha) - \sum_{n=2}^{\infty} (n+1-2\alpha) C(\gamma, n) a_n],$$

which is equivalent to (2.2).

Conversely, if (2.2) holds, we wish to show that $g = f * s_{\gamma}$ is in $S^*[\alpha, \beta]$. For $|z| = r < 1$, we have

$$\begin{aligned} \left| \frac{(zg'/g) - 1}{(zg'/g) + (1-2\alpha)} \right| &= \left| \frac{\sum_{n=2}^{\infty} (n-1) C(\gamma, n) a_n z^{n-1}}{2(1-\alpha) - \sum_{n=2}^{\infty} (n+1-2\alpha) C(\gamma, n) a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} (n-1) C(\gamma, n) a_n}{2(1-\alpha) - \sum_{n=2}^{\infty} (n+1-2\alpha) C(\gamma, n) a_n}. \end{aligned}$$

The function g is in $S^*[\alpha, \beta]$ if the last expression is $\leq \beta$, which is equivalent to (2.2). Hence, $f \in R_{\gamma}[\alpha, \beta]$ and the theorem is proved.

COROLLARY. If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_{\gamma}[\alpha, \beta]$, then $a_n \leq 2\beta(1-\alpha)/[(n-1) + \beta(n+1-2\alpha)]C(\gamma, n)$, $n \geq 2$, with equality for functions of the form

$$f_n(z) = z - 2\beta(1-\alpha)z^n / [(n-1) + \beta(n+1-2\alpha)]C(\gamma, n) .$$

It follows from Theorem 1 that $R_{\gamma}[\alpha, \beta]$ is a closed, convex family. We shall now show that the extreme points of the closed convex hull are those that maximize the coefficients.

THEOREM 2. Set

$$f_1(z) = z \text{ and } f_n(z) = z - 2\beta(1-\alpha)z^n / [(n-1) + \beta(n+1-2\alpha)]C(\gamma, n) , \tag{2.4}$$

$n = 2, 3, \dots$. Then $f \in R_{\gamma}[\alpha, \beta]$, $0 \leq \alpha$, $\gamma < 1$, $0 < \beta \leq 1$, if and only if it can be expressed as $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

PROOF. If $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, then

$$\sum_{n=2}^{\infty} \frac{[(n-1) + \beta(n+1-2\alpha)]C(\gamma, n)}{2\beta(1-\alpha)} \cdot \frac{\lambda_n (2\beta)(1-\alpha)}{[(n-1) + \beta(n+1-2\alpha)]C(\gamma, n)} = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1$$

and $f \in R_{\gamma}[\alpha, \beta]$.

Conversely, if $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_{\gamma}[\alpha, \beta]$, then set

$$\lambda_n = [(n-1) + \beta(n+1-2\alpha)]C(\gamma, n) a_n / 2\beta(1-\alpha), \quad n = 2, 3, \dots, \text{ and set } \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n .$$

We see from Theorem 1 that $\lambda_1 \geq 0$. Since $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, the proof is complete.

3. DISTORTION THEOREMS.

We may now find bounds on the modulus of f and f' for $f \in R_{\gamma}[\alpha, \beta]$.

THEOREM 3. If $f \in R_{\gamma}[\alpha, \beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, and either

$$0 \leq \gamma \leq (2+3\beta-\alpha\beta)/(2+4\beta-2\alpha\beta) \text{ or } r \leq (1+2\beta-\alpha\beta)/(1+3\beta-2\alpha\beta), \text{ then, for } |z| \leq r ,$$

$$\max\{0, r - \beta(1-\alpha)r^2 / [(1+\beta(3-2\alpha))(1-\gamma)]\} \leq |f(z)| \leq r + \beta(1-\alpha)r^2 / [1+\beta(3-2\alpha)](1-\gamma) . \text{ The bounds}$$

are sharp, with extremal function $f_2(z) = z - \beta(1-\alpha)z^2 / [1+\beta(3-2\alpha)](1-\gamma)$.

$$\max\{0, r - \max_n \frac{2\beta(1-\alpha)r^n}{[(n-1)+\beta(n+1-2\alpha)]C(\gamma, n)}\} \leq |f(z)| \leq r + \max_n \frac{2\beta(1-\alpha)r^n}{[(n-1)+\beta(n+1-2\alpha)]C(\gamma, n)} .$$

Under the constraints for γ and r , it suffices to show that

$$\Psi(\alpha, \beta, \gamma, r, n) = 2\beta(1-\alpha)r^n / [(n-1)+\beta(n+1-2\alpha)]C(\gamma, n) \tag{3.1}$$

is a decreasing function of n for $n \geq 2$. From (2.1) we see that

$C(\gamma, n+1) = [(n+1-2\gamma)/n]C(\gamma, n)$ so that $\Psi(\alpha, \beta, \gamma, r, n) \geq \Psi(\alpha, \beta, \gamma, r, n+1)$ if and only if

$$h(\alpha, \beta, \gamma, r, n) = (n+1-2\gamma)[n+\beta(n+2-2\alpha)] - rn[n-1+\beta(n+1-2\alpha)] \geq 0 . \tag{3.2}$$

For α and β fixed, the function h is decreasing in γ and r and increasing in n . Hence, $h(\alpha, \beta, \gamma, r, n) \geq h(\alpha, \beta, (2+3\beta-\alpha\beta)/(2+4\beta-2\alpha\beta), 1, 2) = 0$ for

$0 \leq \gamma \leq (2+3\beta-\alpha\beta)/(2+4\beta-2\alpha\beta)$, $r < 1$, and $n \geq 2$. Similarly,

$h(\alpha, \beta, \gamma, r, n) \geq h(\alpha, \beta, 1, (1+2\beta-\alpha\beta)/(1+3\beta-2\alpha\beta), 2) = 0$ for

$0 \leq \gamma < 1$, $r \leq (1+2\beta-\alpha\beta)/(1+3\beta-2\alpha\beta)$, and $n \geq 2$. Thus $\max_{n \geq 2} \Psi(\alpha, \beta, \gamma, r, n)$ is attained at

$n=2$, and the proof is complete.

As a special case of Theorem 3, we get the result in [3] as a

COROLLARY. If $f \in R_\alpha[\alpha, 1]$, $0 \leq \alpha < 1$, then

$$r - r^2/2(2-\alpha) \leq |f(z)| \leq r + r^2/2(2-\alpha) \quad (|z|=r) .$$

PROOF. When $\beta = 1$, we have $\gamma = \alpha \leq (5-\alpha)/(6-2\alpha)$, so that the first condition in Theorem 3 is satisfied.

REMARK. The function $f_2(z) = 0$ in Theorem 3 when

$z = [1+\beta(3-2\alpha)](1-\gamma)/\beta(1-\alpha)$. Letting $z \rightarrow 1^-$, we thus have

$|f(z)| \geq r - \beta(1-\alpha)r^2/[1+\beta(3-2\alpha)](1-\gamma)$ for all z in E if and only if

$0 \leq \gamma \leq [1+\beta(2-\alpha)]/[1+\beta(3-2\alpha)]$.

Theorem 3 leaves open the question of an upper bound for $|f|$ when $\gamma > (2+3\beta-\alpha\beta)/(2+4\beta-2\alpha\beta)$ and $r > (1+2\beta-\alpha\beta)/(1+3\beta-2\alpha\beta)$. We resolve this with

THEOREM 4. Set $r_{n_0}(\alpha, \beta, \gamma) = (n_0+1-2\gamma)[n_0+\beta(n_0+2-2\alpha)]/n_0[n_0-1+\beta(n_0+1-2\alpha)]$.

If $f \in R_\gamma[\alpha, \beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$,

$$\gamma_0 = \frac{(1+\beta)n_0+\beta(1-\alpha)}{n_0+\beta(n_0+2-2\alpha)} < \gamma \leq \frac{1+(1+\beta)n_0+\beta(2-\alpha)}{1+(1+\beta)n_0+\beta(3-2\alpha)} = \gamma_1 \quad (n_0=2, 3, \dots)$$

and $r_{n_0}(\alpha, \beta, \gamma) < r < 1$, then

$$|f(z)| \leq r + 2\beta(1-\alpha)r^{n_0+1} / [n_0 + \beta(n_0 + 2 - 2\alpha)]C(\gamma, n_0 + 1) \quad (|z|=r) ,$$

with equality for f_{n_0+1} given in (2.4).

PROOF. It suffices to determine when $\Psi(\alpha, \beta, \gamma, r, n)$, defined in (3.1), is maximized for $n = n_0 + 1 > 2$. The function Ψ attains its maximum value at $n = n_0 + 1$ if the function h , defined in (3.2), is negative for $n = n_0$ and positive for $n = n_0 + 1$, which occurs for $r_{n_0}(\alpha, \beta, \gamma) < r < r_{n_0+1}(\alpha, \beta, \gamma)$; however, $r_{n_0}(\alpha, \beta, \gamma) < 1$ if and only if $\gamma \geq \gamma_0$ and $r_{n_0+1}(\alpha, \beta, \gamma) \geq 1$ for $\gamma \leq \gamma_1$. Therefore, $\max_n \psi(\alpha, \beta, \gamma, r, n)$ occurs at $n = n_0 + 1$ for $r_{n_0}(\alpha, \beta, \gamma) < r < 1$ and $\gamma_0 \leq \gamma \leq \gamma_1$, and the proof is complete.

We use similar methods to determine a distortion theorem for f' .

THEOREM 5. If $f \in R_\gamma[\alpha, \beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, and either $0 \leq \gamma \leq 1/2$ or $r \leq (2+4\beta-2\alpha\beta)/(3+9\beta-6\alpha\beta) = r_0$, then

$$1 - 2\beta(1-\alpha)r / [1 + \beta(3-2\alpha)](1-\gamma) \leq |f'(z)| \leq 1 + 2\beta(1-\alpha)r / [1 + \beta(3-2\alpha)](1-\gamma) \text{ for } |z| = r ,$$

with equality when $f_2(z) = z - 2\beta(1-\alpha)z^2 / [1 + \beta(3-2\alpha)](1-\gamma)$.

PROOF. For $A(\alpha, \beta, \gamma, r, n) = 2\beta(1-\alpha)nr^{n-1} / [(n-1) + \beta(n+1-2\alpha)]C(\gamma, n)$ we have, according to Theorem 2,

$$1 - \max_{n > 2} A(\alpha, \beta, \gamma, r, n) \leq |f'(z)| \leq 1 + \max_{n > 2} A(\alpha, \beta, \gamma, r, n) .$$

But A is a decreasing function of n if and only if

$$h_1(\alpha, \beta, \gamma, r, n) = (n+1-2\gamma)[n + \beta(n+2-2\alpha)] - (n+1)r[(n-1) + \beta(n+1-2\alpha)] \geq 0 .$$

Since h_1 is decreasing in r and γ for $\gamma \leq 1/2$ and increasing in n , we have

$$h_1(\alpha, \beta, \gamma, r, n) \geq h_1(\alpha, \beta, 1/2, 1, 2) = 1 - \beta(1-2\alpha) \geq 0$$

for $0 \leq \gamma \leq 1/2$, and

$$h_1(\alpha, \beta, \gamma, r, n) \geq h_1(\alpha, \beta, 1, r_0, 2) = 0 \text{ for } r \leq r_0 .$$

This completes the proof.

REMARK. The theorem is the best possible in that $h_1(\alpha, \beta, 1/2, r, 2) < 0$ for

$r > r_0$ and $A(\alpha, \beta, \gamma, 1, n) > A(\alpha, \beta, \gamma, 1, 2)$ for each fixed $\gamma > 1/2$ and $n = n(\gamma)$ sufficiently large.

4. RADII OF UNIVALENCE, STARLIKENESS, AND CONVEXITY.

As we have seen in Theorem 3, it is possible to have $f(z_0) = 0$, $0 < |z_0| < 1$ for f in $R_\gamma[\alpha, \beta]$, which means that f need not be univalent. We now determine when the family contains only univalent functions.

THEOREM 6. $R_\gamma[\alpha, \beta] \subset S$ if and only if $\gamma \leq 1/2$.

PROOF. Since $z + \sum_{n=2}^{\infty} a_n z^n \in S$ if $\sum_{n=2}^{\infty} n|a_n| \leq 1$, it suffices to show for $\gamma \leq 1/2$ -- according to Theorem 1 -- that

$$[(n-1)+\beta(n+1-2\alpha)]C(\gamma, n)/2\beta(1-\alpha) \geq n \text{ for } n=2, 3, \dots \quad (4.1)$$

But $C(\gamma, n) \geq C(1/2, n) = 1$ for $\gamma \leq 1/2$, so we need only prove (4.1) for $\gamma = 1/2$, which is equivalent to $n[1+\beta-2\beta(1-\alpha)] \geq 1-\beta(1-2\alpha)$. This last inequality is true for $n=2$, and consequently for all $n \geq 2$.

Conversely, since $C(\gamma, n) \rightarrow 0$ for $\gamma > 1/2$, we take $f_n(z)$ defined by (2.4), and note that

$$f'_n(z) = 1 - \frac{2\beta(1-\alpha)nz^{n-1}}{[(n-1)+\beta(n+1-2\alpha)]C(\gamma, n)} = 0$$

for

$$z^{n-1} = [(n-1)+\beta(n+1-2\alpha)]C(\gamma, n)/2\beta(1-\alpha)n,$$

which is less than 1 for n sufficiently large. Thus, $f_n(z)$ is not univalent for $\gamma > 1/2$ and $n = n(\gamma)$ sufficiently large.

Since functions of the form $z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, are starlike if and only if they are univalent [5], we have shown that functions in $R_\gamma[\alpha, \beta]$, $0 \leq \gamma \leq 1/2$, are all starlike. We now determine the largest disk in which such functions are starlike of order δ , $0 \leq \delta < 1$.

THEOREM 7. If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_\gamma[\alpha, \beta]$, $0 \leq \alpha < 1$,

$0 < \beta \leq 1$, $0 \leq \gamma \leq 1/2$, then f is starlike of order δ , $0 \leq \delta < 1$, in the disk

$|z| < r_0$, where

$$r_0 = \inf_n \left[\frac{(1-\delta) [(n-1)+\beta(n+1-2\alpha)] C(\gamma, n)}{2\beta(1-\alpha)(n-\delta)} \right]^{1/(n-1)}$$

with equality for a function of the form (2.4).

PROOF. It suffices to show that $|(zf'/f) - 1| < 1-\delta$ for $|z| < r_0$. But

$$|(zf'/f) - 1| \leq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}} \leq 1 - \delta \quad (|z| = r)$$

if and only if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n r^{n-1} \leq 1. \quad (4.2)$$

In view of Theorem 1, we need only find values of r for which

$$\left(\frac{n-\delta}{1-\delta}\right) r^{n-1} \leq \frac{[(n-1)+\beta(n+1-2\alpha)] C(\gamma, n)}{2\beta(1-\alpha)} \quad (n=2, 3, \dots),$$

which will be true when $r \leq r_0$, and the theorem is proved.

COROLLARY 1. If $f \in R_{\gamma}[\alpha, \beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 \leq \gamma \leq 1/2$, then f is convex of order δ , $0 \leq \delta < 1$ in the disk $|z| < r_1$, where

$$r_1 = \inf_n \left[\frac{(1-\delta) [(n-1)+\beta(n+1-2\alpha)] C(\gamma, n)}{2\beta(1-\alpha)n(n-\delta)} \right]^{1/(n-1)}.$$

PROOF. Since $z + \sum_{n=2}^{\infty} a_n z^n$ is convex of order δ if and only if

$z + \sum_{n=2}^{\infty} n a_n z^{n-1}$ is starlike of order δ , the proof follows that of Theorem 7, with a_n replaced by $n a_n$.

By taking $\delta = 0$ in Theorem 7, we may determine the radius of univalence (and starlikeness) of $R_{\gamma}[\alpha, \beta]$ when $\gamma > 1/2$.

COROLLARY 2. If $f \in R_{\gamma}[\alpha, \beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $1/2 < \gamma < 1$, then f is univalent and starlike for $|z| < r_2$, where

$$r_2 = \inf_n \left[\frac{((n-1)+\beta(n+1-2\alpha))C(\gamma, n)}{2\beta n(1-\alpha)} \right]^{1/(n-1)}$$

5. ORDER OF STARLIKENESS

Since functions in $R_{\gamma}[\alpha, \beta]$, $0 \leq \gamma \leq 1/2$, are starlike, it is of interest to determine the order of starlikeness. We do this in

THEOREM 8. If $f \in R_{\gamma}[\alpha, \beta]$, $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $0 \leq \gamma \leq 1/2$, then f is starlike of order

$$\lambda = \frac{[1+\beta(3-2\alpha)](1-\gamma)-2\beta(1-\alpha)}{[1+\beta(3-2\alpha)](1-\gamma)-\beta(1-\alpha)},$$

with equality for $f(z) = z^{-\beta(1-\alpha)}z^2/[1+\beta(3-2\alpha)](1-\gamma)$.

PROOF. From Theorem 1 and [5], it suffices to show, for

$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in R_{\gamma}[\alpha, \beta]$, that $\sum_{n=2}^{\infty} [(n-1)+\beta(n+1-2\alpha)]C(\gamma, n)a_n/2\beta(1-\alpha) \leq 1$ implies $\sum_{n=2}^{\infty} [(n-\lambda)/(1-\lambda)]a_n \leq 1$. This will be true if

$$g(\alpha, \beta, \gamma, n) = \frac{[(n-1)+\beta(n+1-2\alpha)]C(\gamma, n)(1-\lambda)}{2\beta(1-\alpha)(n-\lambda)} \geq 1 \quad (n=2, 3, \dots).$$

For α and β fixed, g can be shown to be an increasing function of γ , $0 \leq \gamma \leq 1/2$, and an increasing function of n , $n \geq 2$, so that $g(\alpha, \beta, \gamma, n) \geq g(\alpha, \beta, 1/2, 2) = 1$ for $0 \leq \gamma \leq 1/2$ and $n \geq 2$. This completes the proof.

Choosing $\beta = 1$ and $\gamma = \alpha$ in Theorem 8, we get the following result proved in [3] as a

COROLLARY. If $f \in R_{\alpha}[\alpha, 1]$, $0 \leq \alpha \leq 1/2$, then f is starlike of order $(2-2\alpha)/(3-2\alpha)$.

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REFERENCES

1. PADMANABHAN, K.S. On Certain Classes of Starlike Functions In the Unit Disk, J. Indian Math. Soc. 32 (1968), 89-103.
2. RUSCHEWEYH, ST. Linear Operators Between Classes of Prestarlike Functions, Comm. Math. Helv. 52 (1977), 497-509.
3. SILVERMAN, H. and SILVIA, E.M. Prestarlike Functions with Negative Coefficients, Inter. J. Math. & Math. Sci. 2 (1979), 427-439.
4. GUPTA, V.P. and JAIN, P.K. Certain Classes of Univalent Functions with Negative Coefficients, Bull. Aust. Math. Soc. 14 (1976), 409-416.
5. SILVERMAN, H. Univalent Functions With Negative Coefficients, Proc. Amer. Math. Soc. 51 (1975), 109-116.

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