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IDENTITIES INVOLVING ITERATED INTEGRAL TRANSFORMS

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ABSTRACT. A number of identities involving iterated integral transforms are established, making use of the fact that a function which is a linear combination of the Macdonald's function $K_\nu(z)$, where z is a complex variable, is a Fourier kernel.

KEY WORDS AND PHRASES. Macdonald's function, Fourier kernel, Mellin transform, Hankel transform, Laplace transform.
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1. INTRODUCTION.

The object of this note is to establish various identities involving integral operators. The integral operators are the integral transforms with respect to the function $K_\nu(z)$, where $K_\nu(z)$ is the Macdonald's function of order ν and argument z, a complex variable. Some functional relations are deduced, as special cases, which show the inter-relations among more familiar Fourier Sine, Fourier Cosine, and Laplace transforms.

2. THE KERNEL.

Let $\quad y = x^{\frac{1}{2}}K_\nu(\theta x)$, with θ a constant and $|\nu| < 1$.

Then
$$y'' - \frac{\nu^2 - 1/4}{x^2}\, y = \theta^2 y$$

or
$$\left(D^2 - \frac{\nu^2 - 1/4}{x^2}\right) y = \theta^2 y, \qquad D \equiv \frac{d}{dx}\,.$$

Whence
$$\left(D^2 - \frac{\nu^2 - 1/4}{x^2}\right)^k y = \theta^{2k} y, \qquad k = 0,1,2,\ldots\,.$$

Now, if we set $\theta = ie^{i\pi m/k}$, $0 \le m \le 2k-1$, then $$y = x^{\frac{1}{2}}K_\nu(ie^{i\pi m/k}x),$$

satisfies a k-fold Bessel equation:

$$\left(D^2 - \frac{\nu^2 - 1/4}{x^2}\right)^k y = (-1)^k y \tag{2.1}$$

It is not difficult to see that if x is a complex variable, then every point of (2.1) is regular except for a singularity at $x = 0$. Now consider a function of the form

$$G_k(x) = \sum_{m=0}^{2k-1} B_m x^{\frac{1}{2}}K_\nu(ie^{i\pi m/k}x), \qquad |\nu| < 1.$$

The functions are an extension of the functions which were first noted by Guinand. As a special case when $k = 2$, chose the coefficients as

$$B_0 = B_2 = -\frac{1}{\pi}, \qquad B_1 = 0 \text{ and } B_3 = -\frac{2}{\pi}\cos\frac{1}{2}\nu\pi .$$

Then we obtain

$$G_2(x) = -\frac{1}{\pi}x^{\frac{1}{2}}\{K_\nu(ix) + K_\nu(-ix) + 2\cos(\tfrac{1}{2}\nu\pi)K_\nu(x)\}$$
$$= K_\nu(x), \text{ say,} \tag{2.2}$$

and we have

THEOREM 2.1. $y = k_\nu(x)$ is a solution of

$$\left(D^2 - \frac{\nu^2 - 1/4}{x^2}\right)^2 y = y, \qquad 0 < x < \infty,$$

the two-fold Bessel equation.

The function $k_\nu(x)$ is of special interest to us here and we shall develop its properties further.

Using the representations [3]

$$K_\nu(x) = \frac{\pi}{2}\operatorname{cosec}\nu\pi\{I_{-\nu}(x) - I_\nu(x)\}$$

and

$$Y_\nu(x) = \operatorname{cosec}\nu\pi\{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)\},$$

where J_ν, I_ν and Y_ν are the usual Bessel functions, equation (2.2), can be written as

$$k_\nu(x) = x^{\frac{1}{2}}\{\sin\tfrac{1}{2}\nu\pi J_\nu(x) + \cos\tfrac{1}{2}\nu\pi(Y_\nu(x) + \tfrac{2}{\pi}K_\nu(x))\}.$$

These functions arise as kernels in divisor summation formulae of the Hardy-Landau type, involving number theoretic function $\sigma_k(n)$, the number of kth powers of the divisor of n, [4]. If we put $\nu = \pm \frac{1}{2}$, we have

$$k_{\pm\frac{1}{2}}(x) = \pi^{-\frac{1}{2}}(\cos x - \sin x + e^{-x}) ,$$

which obviously satisfies the differential equation

$$D^4 y = y.$$

Next, the Mellin transform of $x^{\frac{1}{2}}K_\nu(\alpha x)$ is given by

$$m\{x^{\frac{1}{2}}K_\nu(\alpha x)\} = \alpha^{-s-\frac{1}{2}}\, 2^{s-3/2}\, \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\nu + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\nu + \tfrac{1}{4}) ,$$

where $Re\ s > |Re\ \nu| - \frac{1}{2}$, [5], whence the Mellin transform of $k_\nu(x)$ defined in (2.2) is given by

$$k^*_\nu(s) = -\frac{2^{s-3/2}}{\pi}\, \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\nu + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\nu + \tfrac{1}{4})$$

$$(i^{-s-\frac{1}{2}} + (-i)^{-s-\frac{1}{2}} + 2\cos \tfrac{1}{2}\nu\pi).$$

On simplifying, we have

$$k^*_\nu(s) = -\frac{2^{s+\frac{1}{2}}}{\pi}\, \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\nu + \tfrac{1}{4})\, \Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\nu + \tfrac{1}{4})$$

$$\cos \tfrac{1}{4}\pi(s + \nu + \tfrac{1}{2}) \cdot \cos \tfrac{1}{4}(s - \nu + \tfrac{1}{2}) ,$$

for $\quad Re\ s > |Re\ \nu| - \frac{1}{2}$.

By repeated use of the relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} ,$$

it is not a difficult matter to see that

$$k^*_\nu(s)k^*_\nu(1-s) = 1.$$

Hence,

THEOREM 2.2. The function $k^*_\nu(x)$ defined by equation (2.2) is a Fourier kernel, [6]. If we define the transformation $T[f]$ by

$$T[f] = \int_0^\infty k_\nu(xt)f(t)dt, \qquad |\nu| < 1 \;, \tag{2.3}$$

then $T^2[f] = f$ and T is involutory since $T^2 = I$, the identity transformation. Making use of the asymptotic expansions of the Macdonald's function $K_\nu(z)$, we have

$$k_\nu(x) = 0(e^{-x}), \qquad x \to \infty$$

and

$$k_\nu(x) = 0(x^{-\nu+\frac{1}{2}}), \qquad x \to 0.$$

Therefore

$$k_\nu(x) \in L^2(0,\infty), \qquad \text{for } |\nu| < 1.$$

Now, if we take $f(x) \in L^2(0,\infty)$, the integral defining the transformation T exists and is in fact absolute convergent. Thus T is a bounded transformation on L^2-space for $|\nu| < 1$.

3. THE OPERATOR.

We shall now define the transformation T in operator notation. Denote the operators K_ν and $K_{\nu,i}$ respectively by

$$K_\nu[f] \equiv K_\nu\{f(x);\, x\} = \int_0^\infty \sqrt{xt}\, K_\nu(xt)f(t)dt$$

and

$$K_{\nu,i}[f] \equiv K_\nu\{f(x);\, ix\} = \int_0^\infty \sqrt{xt}\, K_\nu(ixt)f(t)dt \;,$$

where $f \in L^2(0,\infty)$ and $|\nu| < 1$, with $K_\nu(z)$ being the Macdonald's function. Then the transformation T can be expressed, in operator form, as

$$\begin{aligned} T[f] &= \int_0^\infty k_\nu(xt)f(t)dt \\ &= -\frac{1}{\pi}\left\{\int_0^\infty \sqrt{xt}\, K\ (ixt)f(t)dt + \int_0^\infty \sqrt{xt}\, K(-ixt)f(t)dt \right. \\ &\quad \left. + 2\cos\frac{1}{2}\nu\pi \int_0^\infty \sqrt{xt}\, K_\nu(xt)f(t)dt\right\} \\ &= -\frac{1}{\pi}(K_{\nu,i} + K_{\nu,-i} + 2\cos\frac{1}{2}\nu\pi K_\nu)[f]. \end{aligned}$$

and we can write, symbolically,

$$T \equiv -\frac{1}{\pi}(K_{\nu,i} + K_{\nu,-i} + 2\cos\frac{1}{2}\nu\pi K_\nu) \;.$$

Since $T^2 = I$, the identity transformation, we have

$$\frac{1}{\pi^2}(K_{\nu,i} + K_{\nu,-i} + 2\cos\frac{1}{2}\nu\pi K_\nu)^2 = I$$

or

$$I = \frac{1}{\pi^2}\Big\{K^2_{\nu,i} + K^2_{\nu,-i} + 4\cos^2\frac{1}{2}\nu\pi K^2_\nu + K_{\nu,i}K_{\nu,-i} + K_{\nu,-i}K_{\nu,i}$$
$$+ 2\cos\frac{1}{2}\nu\pi(K_{\nu,i}K_\nu + K_{\nu,-i}K_\nu + K_\nu K_{\nu,i} + K_\nu K_{\nu,-i}\Big\} \tag{3.1}$$

The right-hand side is the linear combination of iterated transformations, which are bounded on L^2-space for $|\nu| < 1$.

Now, using the standard result [5],

$$\int_0^\infty tK_\nu(\alpha t)K_\nu(\beta t)dt = \frac{\pi(\alpha\beta)^{-\nu}}{\sin\nu\pi}\left(\frac{\alpha^{2\nu}-\beta^{2\nu}}{\alpha^2-\beta^2}\right),$$

where $|\nu| < 1$ and $Re(\alpha+\beta) > 0$, we have for example, if $f \in L^2(0,\infty)$,

$$K_{\nu,i}K_\nu[f] = \frac{1}{\pi^2}\int_0^\infty \sqrt{xt}\,K_\nu(ixt)dt\int_0^\infty (ut)^{\frac{1}{2}}K_\nu(ut)f(u)du$$
$$= \frac{1}{\pi^2}\int_0^\infty (xu)^{\frac{1}{2}}f(u)du\int_0^\infty tK_\nu(ixt)K_\nu(ut)dt$$
$$= \frac{-(i)^{-\nu}}{\pi\sin\nu\pi}\int_0^\infty (xu)^{\frac{1}{2}-\nu}f(u)\frac{(ix)^{2\nu}-u^{2\nu}}{x^2+u^2}du$$
$$= -K_\nu K_{\nu,-i}[f].$$

The change of order of integration can be justified by absolute convergence. Thus, we obtain our first identity

$$K_{\nu,i}K_\nu + K_\nu K_{\nu,-i} = 0 \tag{3.2}$$

The identity given by (3.2) can alternatively be established by making use of the Mellin transform theory. That is, the Mellin transform of the iterated operator $K_{\nu,i}K_\nu[f]$, is given formally by

$$m\{K_{\nu,i}K_\nu[f] = m\{-\frac{1}{\pi}x^{\frac{1}{2}}K_\nu(ix);\ s\}m\{-\frac{1}{\pi}x^{\frac{1}{2}}K_\nu(x);\ 1-s\}f^*(s)$$
$$= \frac{i^{-s-\frac{1}{2}}}{4\sin\frac{1}{2}\pi(s-\nu+\frac{1}{2})\sin\frac{1}{2}\pi(s+\nu+\frac{1}{2})}f^*(s),$$

where $f^*(s)$ denotes the Mellin transform of $f(x)$. Also,

$$m\{K_\nu K_{\nu,-i}[f]\} = m\{-\frac{1}{\pi}x^{\frac{1}{2}}K_\nu(x);\ s\}m\{-\frac{1}{\pi}x^{\frac{1}{2}}K_\nu(-ix);\ 1-s\}f^*(s)$$
$$= \frac{(-i)^{s-3/2}}{4\sin\frac{1}{2}\pi(s-\nu+\frac{1}{2})\sin\frac{1}{2}\pi(s+\nu+\frac{1}{2})}.$$

Then $$m\{(K_{\nu,i}K_\nu + K_\nu K_{\nu,-i})[f]\} = 0,$$

implying that

$$K_{\nu,i}K_\nu + K_\nu K_{\nu,-i} = 0,$$

as shown above. Similarly, one can show that

$$K_\nu K_{\nu,i} + K_{\nu,-i}K_\nu = 0, \tag{3.3}$$

a sort of conjugate of the identity in (3.2). Consider the representation

$$K_\nu(ix) = \frac{\pi}{2\sin\nu\pi}(e^{-i\frac{1}{2}\nu\pi}J_{-\nu}(x) - e^{i\frac{1}{2}\nu\pi}J_\nu(x));$$

then

$$K_{\nu,i}[f] = \int_0^\infty (xt)^{\frac{1}{2}}K_\nu(ixt)f(t)\,dt, \qquad |\nu| < 1$$

$$= \frac{\pi}{2\sin\nu\pi}\left\{e^{-i\frac{1}{2}\nu\pi}\int_0^\infty (xt)^{\frac{1}{2}}J_{-\nu}(xt)f(t)\,dt - e^{i\frac{1}{2}\nu\pi}\int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)f(t)\,dt\right\}$$

$$= \frac{\pi}{2\sin\nu\pi}\left\{e^{-i\frac{1}{2}\nu\pi}H_{-\nu}[f] - e^{i\frac{1}{2}\nu\pi}H_\nu[f]\right\},$$

where H_ν denotes the Hankel transform operator of order ν. Thus, in operator form,

$$K_{\nu,i} = \frac{\pi}{2\sin\nu\pi}e^{-i\frac{1}{2}\nu\pi}H_{-\nu} - \frac{\pi}{2\sin\nu\pi}e^{i\frac{1}{2}\nu\pi}H_\nu \tag{3.4}$$

and similarly

$$K_{\nu,-i} = \frac{\pi}{2\sin\nu\pi}e^{i\frac{1}{2}\nu\pi}H_{-\nu} - \frac{\pi}{2\sin\nu\pi}e^{-i\frac{1}{2}\nu\pi}H_\nu .$$

Now, after substituting for $K_{\nu,i}$ and $K_{\nu,-i}$ in (3.2) and rearranging, we have

$$e^{-i\frac{1}{2}\nu\pi}(H_{-\nu}K_\nu - K_\nu H_\nu) = e^{i\frac{1}{2}\nu\pi}(H_\nu K_\nu - K_\nu H_{-\nu}).$$

By comparing the real and imaginary parts and solving, we obtain the identities

$$K_\nu H_\nu = H_{-\nu}K_\nu \tag{3.5}$$

and $$H_\nu K_\nu = K_\nu H_{-\nu} \tag{3.6}$$

Now,

$$H_{\frac{1}{2}}[f] = \int_0^\infty (xt)^{\frac{1}{2}} J_{\frac{1}{2}}(xt) f(t)\, dt$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \sin(xt) f(t)\, dt$$

$$= S[f],$$

and similarly,

$$H_{-\frac{1}{2}}[f] = C[f],$$

where S and C are the usual Fourier since and cosine transform repsectively.

Also,

$$K_{\frac{1}{2}}[f] = \int_0^\infty (xt)^{\frac{1}{2}} K_{\frac{1}{2}}(xt) f(t)\, dt$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-xt} f(t)\, dt$$

$$= \left(\frac{\pi}{2}\right)^{\frac{1}{2}} L[f],$$

where $L[f]$ is the Laplace transform.

Hence, setting $\nu = \pm\frac{1}{2}$ in (3.4), we obtain the relations,

$$LS = CL$$

and
$$LC = SL. \tag{3.7}$$

Incidently, a general relation involving the operators K_ν and H_μ can be established:

$$H_\mu K_\nu = \operatorname{cosec} \nu\pi \{\sin \tfrac{1}{2}(\nu-\mu)\pi K_\mu H_\nu + \sin \tfrac{1}{2}(\nu+\mu)\pi K_\mu H_{-\nu} \tag{3.8}$$

On setting $\mu = \pm\nu$, (3.8) yields the identities (3.5) and (3.6). Next, from the representation (3.4), we have symbolically

$$K^2_{\nu,i} = \left(\frac{\pi}{2\sin \nu\pi}\right)^2 \left(e^{-i\frac{1}{2}\nu\pi} H_{-\nu} - e^{i\frac{1}{2}\nu\pi} H_\nu\right)^2$$

$$= \left(\frac{\pi}{2\sin \nu\pi}\right)^2 (2\cos \nu\pi I - H_{-\nu}H_\nu - H_\nu H_{-\nu}) \tag{3.9}$$

since $H^2_{-\nu} = H^2_\nu = I$, the identity operator, with H_ν being the Hankel transform.

Similarly, one can show that

$$K^2_{\nu,\bar{i}} = K^2_{\nu,-i}\ . \tag{3.10}$$

And, in the same vein, we have

$$K_{\nu,i}K_{\nu,-i} + K_{\nu,-i}K_{\nu,i} = 2\left(\frac{\pi}{2\sin\nu\pi}\right)^2\left\{2I - \cos\nu\pi(H_\nu H_{-\nu} + H_{-\nu}H_\nu)\right\} \tag{3.11}$$

Now, going back to equation (3.1) and using the results given by (3.2), (3.3), and (3.9) in (3.11) and simplifying, we have finally, for $|\nu| < 1$,

$$H_\nu H_{-\nu} + H_{-\nu}H_\nu - \left(\frac{2}{\pi}\sin\nu\pi\right)^2 K_\nu^2 = 2\cos\nu\pi\, I. \tag{3.12}$$

An interesting relationship can be established by putting $\nu = \pm\frac{1}{2}$ in (3.12). Then

$$H_{\frac{1}{2}}H_{-\frac{1}{2}} + H_{-\frac{1}{2}}H_{\frac{1}{2}} - \frac{4}{\pi^2}\; K_{\frac{1}{2}}^2 = 0$$

or

$$SC + CS = \frac{2}{\pi}L^2 \;, \tag{3.13}$$

where S, C, and L denote the Fourier sine, Fourier consine and Laplace transforms respectively.

From (3.9) and (3.12), one can establish the identity

$$K_\nu^2 = -K_{\nu,i}^2 = -K_{\nu,-i}^2 \;, \tag{3.14}$$

which, on setting $\nu = \pm\frac{1}{2}$, yields (3.13).

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