

THE n -DIMENSIONAL DISTRIBUTIONAL MELLIN TRANSFORMATION

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ABSTRACT. The n -dimensional distributional Mellin transformation is developed using the testing function space $M_{C,d}$ and its dual $M'_{C,d}$. The standard theorems on analyticity, uniqueness and continuity are proved. A necessary and sufficient condition for a function to be an n -dimensional Mellin transformation is proved by the help of a boundedness property for distribution in $M'_{C,d}$. Some operational transform formulas are also introduced.

KEY WORDS AND PHRASES. *Distributional Mellin Transformation, Distributions, and Test function spaces.*

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1. INTRODUCTION.

The Mellin transformation was previously extended to certain generalized functions by Zemanian [1] and Fung Kang [2]. In the present paper, we develop the n -dimensional distributional Mellin transformation.

For the sake of brevity, we shall use the following notations. R^n and C^n are respectively real and complex n -dimensional euclidean spaces. The symbols z and s stand for elements of C^n representing the n -triples (z_1, z_2, \dots, z_n) and (s_1, s_2, \dots, s_n) respectively. We take $x \in R^n$, $t \in R^n$, $\sigma \in R^n$, $\omega \in R^n$ and $s = \sigma + i\omega \in C^n$. A function on a subset of R^n shall be denoted by $h(x) = h(x_1, x_2, \dots, x_n)$. By $[x]$ we mean the product x_1, x_2, \dots, x_n . Thus, $[x^s] = x_1^{s_1}, x_2^{s_2}, \dots, x_n^{s_n}$ where $s = \{s_1, s_2, \dots, s_n\}$ and $[e^{-st}] = \exp(-s_1 t_1 - \dots - s_n t_n)$. By $\log x$ we mean

$\{\log x_1, \dots, \log x_n\}$ and, by xt , we mean $\{x_1 t_1, x_2 t_2, \dots, x_n t_n\}$. Also, $x^s = \{x_1^{s_1}, \dots, x_n^{s_n}\}$ and $e^{-st} = \{e^{-s_1 t_1}, \dots, e^{-s_n t_n}\}$. The notation $x \leq y$ and $x < y$ mean respectively $x_\nu \leq y_\nu$ and $x_\nu < y_\nu$ ($\nu = 1, 2, \dots, n$). The letters k and m shall denote non-negative integers in \mathbb{R}^n , i.e., k_ν and m_ν are non-negative integers. Letting $k = k_1 + k_2 + \dots + k_n$, D_x^k shall denote $\frac{\sigma^k}{\sigma x_1^{k_1} \sigma x_2^{k_2} \dots \sigma x_n^{k_n}}$.

By a smooth function we mean a function that possesses partial derivatives of all orders at all points of its domain.

2. THE TESTING FUNCTION SPACE $M_{c,d}$.

Let \mathbb{R}_+^n denote the open domain $0 < x < \infty$. We define $\eta_{c,d}(x)$ as the product function $\prod_{\nu=1}^n \eta_{c_\nu, d_\nu}(x_\nu)$ where $\eta_{c_\nu, d_\nu}(x_\nu) = \begin{cases} x_\nu^{-c_\nu} & \text{if } 0 < x_\nu < 1/e \\ x_\nu^{-d_\nu} & \text{if } e < x_\nu < \infty \end{cases}$.

In fact, $M_{c,d}$ is the linear space of all smooth functions $f(x)$ defined on \mathbb{R}_+^n with values in C^1 , which satisfy the following set of inequalities.

For each non-negative integer k ,

$$\left| \eta_{c,d}(x) [x^{k+1}] D_x^k f(x) \right| \leq Q_k, \quad 0 < x < \infty. \tag{2.1}$$

Q_k denotes constants which depend upon the choices of k and f .

Any smooth function, whose support is contained in \mathbb{R}_+^n , is in $M_{c,d}$. Other members of $M_{c,d}$ are $[x^{s-1}]$ for $c \leq \text{Re } s \leq d$ and $[(\log x)^k x^{s-1}]$ for $c < \text{Re } s < d$.

μ_ν represents a seminorm defined by

$$\mu_\nu = \mu_\nu(f) = \max_{0 \leq |k| \leq \nu} \sup_x \left| \eta_{c,d}(x) [x^{k-1}] D_x^k f(x) \right|. \tag{2.2}$$

Of course, the collection $\{\mu_\nu\}$ is a multinorm, being a separating collection of seminorms. Thus we can assign to $M_{c,d}$ the topology generated by $\{\mu_\nu\}$.

A sequence $\{f_\nu\}_{\nu=1}^\infty$ is a Cauchy sequence in $M_{c,d}$ if and only if each $f_\nu \in M_{c,d}$ and, for each fixed k , the functions $\eta_{c,d}(x) [x^{k+1}] D_x^k f_\nu(x)$ converges uniformly on \mathbb{R}_+^n as $\nu \rightarrow \infty$. Hence, $M_{c,d}$ is sequentially complete.

THEOREM 2.1. The mapping

$$f(x) \rightarrow [e^{-p}] f(e^{-p}) = g(p) \tag{2.3}$$

is an isomorphism from $M_{c,d}$ into $L_{c,d}$ where $L_{c,d}$ denotes the testing function space

defined by Sinha [3].

The inverse mapping is given by

$$g(p) \rightarrow [x^{-1}] f(-\log x) = f(x) \tag{2.4}$$

PROOF. The proof of this theorem is easy and is therefore omitted.

3. THE DUAL SPACE $M'_{c,d}$.

$M'_{c,d}$ is the dual space of $M_{c,d}$. Multiplication by a complex number, equality, and addition are defined in the usual way. In fact, $M'_{c,d}$ is a linear space over C^1 . By $\langle h, f \rangle$ we mean a number that $h \in M'_{c,d}$ assigns to $f \in M_{c,d}$. If the support (Miller [4], §1.6) of a distribution h is contained in a compact subset of R_+^n , then $h \in M'_{c,d}$, $c, d \in R^n$ with $c < d$. Also, every member of $M'_{c,d}$ is a distribution on R_+^n .

Let us define a (weak) topology for $M'_{c,d}$ by using the following separating set of seminorms. For every $f \in M_{c,d}$, we define a seminorm $\zeta_f(h)$ on $M'_{c,d}$ by

$$\zeta_f(h) = |\langle h, f \rangle|, \quad (h \in M'_{c,d}).$$

In fact, a sequence $\{h_\nu\}_{\nu=1}^\infty$ ($h \in M'_{c,d}$) is a Cauchy sequence in $M'_{c,d}$ if and only if, for all $f \in M_{c,d}$, the numerical sequence $\{\langle h_\nu, f \rangle\}_{\nu=1}^\infty$ converges.

We can easily prove that $M'_{c,d}$ is sequentially complete.

In view of Theorem 1, we can relate to each $h(x) \in M'_{c,d}$ a distribution $h(e^{-p}) \in L'_{c,d}$ (see [3]) by

$$\langle h(e^{-p}), g(p) \rangle = \langle h(x), f(x) \rangle. \tag{3.1}$$

Conversly, if $\psi(p) \in L'_{c,d}$, then $\psi(-\log x) \in M'_{c,d}$ is given by

$$\langle \psi(-\log x), f(x) \rangle = \langle \psi(p), g(p) \rangle. \tag{3.2}$$

Using (3.1) and (3.2), we can easily have the following theorem:

THEOREM 3.1. The mapping $h(x) \rightarrow h(e^{-p})$ defined by (3.1), is an isomorphism from $M'_{c,d}$ onto $L'_{c,d}$. The inverse mapping is given by (3.2).

4. THE n-DIMENSIONAL DISTRIBUTIONAL MELLIN TRANSFORMATION M.

DEFINITION. We define the n-dimensional distributional Mellin transformation Mh as the function $H(s)$ on Ω_h into C^1 by

$$(Mh)(s) = H(s) = \langle h(x), [x^{s-1}] \rangle \quad \text{for } s \in \Omega_h, \tag{4.1}$$

where Ω_h is the tube of definition of the n-dimensional distributional Laplace

transformation (see [3]).

In fact, the R.H.S. of (4.1) has a meaning because the application of $h \in M'_{c,d}$ to $[x^{s-1}] \in M_{c,d}$.

Setting $g(p) = [e^{-sp}]$ and $f(x) = [x^{-1}] g(-\log x) = [x^{s-1}]$ and using Theorem 2.1, we can have the following theorem:

THEOREM 4.1. The distribution $h(x)$ is n -dimensional Mellin transformable if $h(e^{-p})$ is n -dimensional Laplace transformable. In such a case, $Mh(x) = H(s) = Lh(e^{-p})$ for ever $s \in \Omega_h$.

Using Theorems 2.1 and 3.1, we can have the following theorems:

THEOREM 4.2. (The Analyticity Theorem). If $Mh = H(s)$ for $s \in \Omega_h$, then $H(s)$ is analytic on Ω_h and

$$\frac{\partial H}{\partial s_\nu} = \langle (\log x_\nu) h(p), [x^s] \rangle, \quad s \in \Omega_h. \tag{4.2}$$

The proof is analogous to that given in [3].

THEOREM 4.3. (The Uniqueness Theorem). If $Mh = H(s)$ for $s \in \Omega_h$ and $Mg = G(s)$ for $s \in \Omega_g$, if $\Omega_h \cap \Omega_g$ is non-void, and if $H(s) = G(s)$ for $s \in \Omega_h \cap \Omega_g$, then $h=g$ in the same sense of equality in $M'_{c,d}$ where $c,d \in \Omega_h$ and $c < d$. The proof is analogous to that given in [3].

THEOREM 4.4. (The Continuity Theorem). If $\{h_\nu\}_{\nu=1}^\infty$ converges in $M'_{c,d}$ to h for some $c,d \in R_+^n$ ($c < d$) and if $Mh_\nu = H_\nu(s)$, then $Lh = H(s)$ exists for at least $c < \text{Re } s < d$ and $\{H_\nu(s)\}_{\nu=1}^\infty$ converges pointwise in the tube of definition $c < \text{Re } s < d$ to $H(s)$.

PROOF. Since $[x^s]$ is in $M_{c,d}$ for each s satisfying $c < \text{Re } s < d$, the theorem follows from the definition of convergence in $M'_{c,d}$ and the fact that $M'_{c,d}$ is sequentially complete.

5. A BOUNDEDNESS PROPERTY FOR DISTRIBUTIONS IN $M'_{c,d}$.

For each $h \in M'_{c,d}$, there exists a non-negative integer $r \in R_+^1$ and a positive constant $c \in R_+^1$ such that, for all ψ in $M_{c,d}$,

$$|\langle h, \psi \rangle| \leq c\mu(\psi). \tag{5.1}$$

6. A NECESSARY AND SUFFICIENT CONDITION FOR $M(s)$ TO BE AN n -DIMENSIONAL MELLIN TRANSFORM.

A necessary and sufficient condition for a function $M(s)$ to be the n -dimensional Mellin transform of a distribution h is that there be a tube $c < \operatorname{Re} s < d$ ($c < d$) on which $M(s)$ is analytic and bounded when

$$|M(s)| \leq P(|s|) \quad (6.1)$$

where $P(|s|)$ is a polynomial in $|s|$.

It can be easily proved by using the boundedness property of Section 5 and (Bochner [6], Theorem 60, p. 242 and §4, p. 244).

7. SOME OPERATIONAL TRANSFORM FORMULAS FOR THE n -DIMENSIONAL DISTRIBUTIONAL MELLIN TRANSFORMATION.

Let us suppose that $Mh(p) = H(s)$ for $s \in \Omega_h$ and $p \in \mathbb{R}_+^n$, $\alpha \in \mathbb{C}^n$. We can easily have the following operational transform formulas (Using Theorem 4):

$$(11) \quad M D_p^k h(p) = s^k H(s), \quad s \in \Omega_h,$$

$$(12) \quad M\{[x^\alpha]h\} = H(s + \alpha), \quad s + \alpha \in \Omega_h,$$

$$(13) \quad Mh(\log x) = H(-s), \quad -s \in \Omega_h,$$

$$(14) \quad Mh\{\tau(-\log x)\} = [\tau^{-1}] H(s/\tau), \quad s/\tau \in \Omega_h, \quad \tau > 0.$$

Also, by using Theorem 5, we can have

$$(15) \quad M\{(-\log x)^k h(-\log x)\} = (-)^{|k|} D_s^k H(s), \quad s \in \Omega_h.$$

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