

ON CONTACT CR-SUBMANIFOLDS OF SASAKIAN MANIFOLDS

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ABSTRACT: Recently, K.Yano and M.Kon [5] have introduced the notion of a contact CR-submanifold of a Sasakian manifold which is closely similar to the one of a CR-submanifold of a Kaehlerian manifold defined by A.Bejancu [1].

In this paper, we shall obtain some fundamental properties of contact CR-submanifolds of a Sasakian manifold. Next, we shall calculate the length of the second fundamental form of a contact CR-product of a Sasakian space form (THEOREM 7.4). At last, we shall prove that a totally umbilical contact CR-submanifold satisfying certain conditions is totally geodesic in the ambient manifold (THEOREM 8.1).

KEY WORDS AND PHRASES : *Kaehlerian manifold, Sasakian space form, contact CR-product.*

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1. INTRODUCTION.

This paper is directed to specialist readers with background in the area and appreciative of its relation of this area of study.

Let \tilde{M} be a $(2n + 1)$ -dimensional Sasakian manifold with structure tensors $(\phi, \xi, \eta, \langle, \rangle)$ [4] and let M be an m -dimensional Riemannian manifold isometrically immersed in \tilde{M} and let \langle, \rangle be the induced metric on M . Let ∇ and $\tilde{\nabla}$ be the covariant differentiations on M and \tilde{M} , respectively. Then the Gauss and Weingarten's formulas for M are respectively given by

$$\tilde{\nabla}_U V = \nabla_U V + \sigma(U, V), \quad (1.1)$$

$$\tilde{\nabla}_U \lambda = -A_\lambda U + \nabla_U^\perp \lambda \quad (1.2)$$

for any vector fields U, V tangent to M and any vector field λ normal to M , where σ denotes the second fundamental form and ∇^\perp is the normal connection. The second fundamental tensor A_λ is related to σ by

$$\langle A_\lambda U, V \rangle = \langle \sigma(U, V), \lambda \rangle. \quad (1.3)$$

The mean curvature vector H is defined by

$$H = \frac{1}{m} \text{trace } \sigma. \quad (1.4)$$

The submanifold M is called a minimal submanifold of \tilde{M} if $H = 0$ and M is called a totally geodesic submanifold of \tilde{M} if $\sigma = 0$.

For any vector field U tangent to M , we put

$$\phi U = PU + FU, \quad (1.5)$$

where PU and FU are the tangential and the normal components of ϕU , respectively. Then P is an endomorphism of the tangent bundle TM of M and F is a normal-bundle-valued 1-form of TM .

For any vector field λ normal to M , we put

$$\phi \lambda = t\lambda + f\lambda, \quad (1.6)$$

where $t\lambda$ and $f\lambda$ are the tangential and the normal components of $\phi \lambda$, respectively.

Then f is an endomorphism of the normal bundle $T^\perp M$ of M and t is a tangent-bundle-valued 1-form of $T^\perp M$.

We put

$$\xi = \xi_1 + \xi_2, \quad (1.7)$$

where ξ_1 and ξ_2 are the tangential and the normal components of ξ , respectively.

Then we can put

$$\eta = \eta_1 + \eta_2, \quad (1.8)$$

where $\eta_1(U) = \langle \xi_1, U \rangle$ and $\eta_2(\lambda) = \langle \xi_2, \lambda \rangle$ for any vector field U tangent to M and any vector field λ normal to M .

By virtue of (1.5), (1.6), (1.7) and (1.8), we get

$$P^2U + tFU = -U + \eta_1(U)\xi_1, \tag{1.9}$$

$$FPU + fFU = \eta_1(U)\xi_2, \tag{1.10}$$

$$\eta_2(\lambda)\xi_1 = Pt\lambda + tf\lambda, \tag{1.11}$$

$$Ft\lambda + f^2\lambda = -\lambda + \eta_2(\lambda)\xi_2 \tag{1.12}$$

for any vector field U tangent to M and any vector field λ normal to M .

Let $\tilde{M}(k)$ be a Sasakian space form with constant ϕ -holomorphic sectional curvature k . Then the curvature tensor \tilde{R} of $\tilde{M}(k)$ is given by

$$\begin{aligned} \tilde{R}(X, Y)Z = & \frac{k+3}{4} \{ \langle Y, Z \rangle X - \langle X, Z \rangle Y \} + \frac{k-1}{4} \{ \eta(X)\langle Y, Z \rangle \xi - \eta(Y)\langle X, Z \rangle \xi \\ & + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - \langle \phi Y, Z \rangle \phi X + \langle \phi X, Z \rangle \phi Y + 2\langle \phi X, Y \rangle \phi Z \} \end{aligned} \tag{1.13}$$

for any vector fields X, Y and Z in $\tilde{M}(k)$ [3].

For the second fundamental form σ , we define the covariant differentiation $\bar{\nabla}$ with respect to the connection on $TM \oplus T^\perp M$ by

$$(\bar{\nabla}_V \sigma)(V, W) = \nabla_V^{\perp}(\sigma(V, W)) - \sigma(\nabla_V V, W) - \sigma(V, \nabla_V W) \tag{1.14}$$

for any vector fields U, V and W tangent to M . We denote R the curvature tensor associated with ∇ . Then the equations of Gauss and Codazzi are respectively given by

$$\tilde{R}(U, V; W, Z) = R(U, V; W, Z) + \langle \sigma(U, W), \sigma(V, Z) \rangle - \langle \sigma(U, Z), \sigma(V, W) \rangle, \tag{1.15}$$

$$(\tilde{R}(U, V)W)^\perp = (\bar{\nabla}_V \sigma)(U, W) - (\bar{\nabla}_U \sigma)(V, W) \tag{1.16}$$

for any vector fields U, V, W and Z tangent to M , where $\tilde{R}(U, V; W, Z) = \langle \tilde{R}(U, V)W, Z \rangle$ and $(\tilde{R}(U, V)W)^\perp$ denotes the normal component of $\tilde{R}(U, V)W$.

2. CONTACT CR-SUBMANIFOLDS OF A SASAKIAN MANIFOLD.

DEFINITION 2.1: A submanifold M of a Sasakian manifold \tilde{M} with structure tensors $(\phi, \xi, \eta, \langle, \rangle)$ is called a contact CR-submanifold if there is a differentiable distribution $D: x \longrightarrow D_x \subseteq T_x M$ on M satisfying the following conditions:

- (i) $\xi \in D$,
- (ii) $\phi D_x \subset T_x M$ for each x in M ,
- (iii) the complementary orthogonal distribution $D^\perp: x \longrightarrow D_x^\perp \subset T_x M$

satisfies $\phi D_x^\perp \subseteq T_x^\perp M$ for each point x in M .

Let M be a contact CR-submanifold of a Sasakian manifold \tilde{M} . Then $\xi \in D \subset TM$,

so, the equations (1.9), (1.10), (1.11) and (1.12) can be written as

$$P^2U + tFU = -U + \eta(U)\xi, \quad (2.1)$$

$$FPU + fFU = 0, \quad (2.2)$$

$$Pt\lambda + tf\lambda = 0, \quad (2.3)$$

$$Ft\lambda + f^2\lambda = -\lambda \quad (2.4)$$

for any vector field U tangent to M and any vector field λ normal to M , respectively.

By virtue of (1.5), we have

PROPOSITION 2.1: In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , in order to a vector field U tangent to M belong to D it is necessary and sufficient that $FU = 0$.

Taking account of (2.1) and PROPOSITION 2.1, we have

$$P^2X = -X + \eta(X)\xi \quad (2.5)$$

for any X in D and we have from (1.6)

$$P\xi = 0. \quad (2.6)$$

Furthermore, we obtain

$$\langle PX, PY \rangle = \langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y) \quad (2.7)$$

for any X and Y in D . Thus we have

PROPOSITION 2.2: In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , the distribution D has an almost contact metric structure $(P, \xi, \eta, \langle, \rangle)$ and hence $\dim D_x = \text{odd}$.

We denote by Π the complementary orthogonal subbundle of ϕD^\perp in $T^\perp M$. Then we have

$$T^\perp M = \phi D^\perp \oplus \Pi, \quad \phi D^\perp \perp \Pi. \quad (2.8)$$

Thus we have

PROPOSITION 2.3: For a contact CR -submanifold M of a Sasakian manifold \tilde{M} , the subbundle Π has an almost complex structure f and hence $\dim \Pi_x = \text{even}$.

3. BASIC PROPERTIES.

Let M be a contact CR -submanifold of a Sasakian manifold M . Then we have

$$\phi(\nabla_U Z + \sigma(U, Z)) = -A_{\phi Z} U + \nabla_U(\phi Z) - \langle U, Z \rangle \xi \quad (3.1)$$

for any vector field U tangent to M and Z in D^\perp . From (3.1), we get

$$\langle \nabla_U Z, \phi X \rangle = \langle A_{\phi Z} U, X \rangle + \eta(X) \langle U, Z \rangle \quad (3.2)$$

for any vector field U tangent to M , X in D and Z in D^\perp . In (3.2), if we put $X = \phi X$, then (3.2) means

$$\langle \nabla_U Z, X \rangle = \langle \phi A_{\phi Z} U, X \rangle + \eta(X) \langle \nabla_U Z, \xi \rangle \quad (3.3)$$

for any vector field U tangent to M , X in D and Z in D^\perp . By virtue of (3.2), we obtain

$$\langle A_{\phi Z} X, \xi \rangle = 0 \quad (\text{or equivalently } \langle \sigma(X, \xi), \phi Z \rangle = 0) \quad (3.4)$$

for any X in D and Z in D^\perp .

On the other hand, we have

$$\langle \sigma(X, \xi), \lambda \rangle = \langle \tilde{\nabla}_X \xi - \nabla_X \xi, \lambda \rangle = \langle \phi X, \lambda \rangle = 0$$

for any X in D and λ in Π . Thus we have from (3.4) and the above equation $\sigma(X, \xi) = 0$ for any X in D . So, we have from (1.7) and the last equation

$$\nabla_X \xi = PX \quad (3.5)$$

for any X in D . Thus we have

PROPOSITION 3.1: In a contact CR-submanifold M of a Sasakian manifold \tilde{M} , the distribution D has a K -contact metric structure $(P, \xi, \eta, \langle, \rangle)$.

In (3.1), if we put $U = W \in D^\perp$, then the equation (3.1) can be written as

$$\phi(\nabla_W Z + \sigma(Z, W)) = -A_{\phi Z} W + \nabla_W^\perp \phi Z - \langle W, Z \rangle \xi,$$

from which

$$\phi([Z, W]) = A_{\phi Z} W - A_{\phi W} Z + \nabla_Z^\perp \phi W - \nabla_W^\perp \phi Z, \quad (3.6)$$

where $[Z, W] = \nabla_Z W - \nabla_W Z$.

LEMMA 3.2: In a contact CR-submanifold M of a Sasakian manifold \tilde{M} , we have

$$\nabla_W^\perp \phi Z - \nabla_Z^\perp \phi W \in \phi D^\perp \quad (3.7)$$

for any Z and W in D^\perp .

PROOF: For any Z, W in D^\perp and λ in Π , we obtain

$$\begin{aligned} \langle \nabla_W^\perp \phi Z - \nabla_Z^\perp \phi W, \lambda \rangle &= \langle \tilde{\nabla}_W \phi Z - \tilde{\nabla}_Z \phi W, \lambda \rangle = \langle \phi(\tilde{\nabla}_W Z - \tilde{\nabla}_Z W) + (\tilde{\nabla}_W \phi)Z - (\tilde{\nabla}_Z \phi)W, \lambda \rangle \\ &= \langle \phi(\nabla_W Z - \nabla_Z W), \lambda \rangle = \langle \nabla_Z W - \nabla_W Z, \phi \lambda \rangle = 0. \end{aligned}$$

On the other hand, we can easily have

$$A_{\phi Z} W = A_{\phi W} Z \tag{3.8}$$

for any Z and W in D^\perp . In fact, for any vector field U tangent to M and Z and W in D^\perp , we have from (3.1)

$$\begin{aligned} \langle \phi(\nabla_U Z + \sigma(U, Z)), W \rangle &= \langle \phi \nabla_U Z, W \rangle + \langle \phi \sigma(U, Z), W \rangle = -\langle \sigma(U, Z), \phi W \rangle \\ &= -\langle A_{\phi W} Z, U \rangle = -\langle A_{\phi Z} W, U \rangle, \end{aligned}$$

from which, we have (3.8).

By virtue of (3.6) and (3.8) and LEMMA 3.2, we have

PROPOSITION 3.3: In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , the distribution D^\perp is integrable.

For any X in D and λ in Π , we have

$$A_\lambda \phi X = -A_{\phi \lambda} X. \tag{3.9}$$

Next, we assume that the distribution D is integrable. Then for any X and Y in D , $\phi[X, Y]$ is an element of D , that is, $\phi[X, Y] \in TM$. Since we have

$$\phi[X, Y] = \phi(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) = \{\nabla_X \phi Y - \nabla_Y \phi X + \eta(X)Y - \eta(Y)X\} + \{\sigma(X, \phi Y) - \sigma(\phi X, Y)\},$$

we get $\sigma(X, \phi Y) = \sigma(\phi X, Y)$. From which we obtain

$$\langle \sigma(X, \phi Y), \phi Z \rangle = \langle \sigma(\phi X, Y), \phi Z \rangle \tag{3.10}$$

for any X and Y in D and Z in D^\perp .

Conversely, if (3.10) is satisfied, we can easily show that the distribution D is integrable. Thus we have

PROPOSITION 3.4: In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , the distribution D is integrable if and only if the equation (3.10) is satisfied.

Next, we can prove

PROPOSITION 3.5: In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , the vector field ξ is parallel along D^\perp .

PROOF: For any Z in D^\perp , we have

$$\phi Z = \nabla_Z \xi + \sigma(Z, \xi).$$

Since the vector field Z is an element of D^\perp , ϕZ is in $T^\perp M$. Thus we have from the above equation $\nabla_Z \xi = 0$, that is, the vector field ξ is parallel along any vector field in D^\perp .

4. SOME COVARIANT DIFFERENTIATIONS.

DEFINITION 4.1: In a contact CR-submanifold M of a Sasakian manifold \tilde{M} , we define

$$(\bar{\nabla}_U P)V = \nabla_U(PV) - P\nabla_U V, \quad (4.1)$$

$$(\bar{\nabla}_U F)V = \nabla_U^{\perp}(FV) - F\nabla_U V, \quad (4.2)$$

$$(\bar{\nabla}_U t)\lambda = \nabla_U(t\lambda) - t\nabla_U^{\perp}\lambda, \quad (4.3)$$

$$(\bar{\nabla}_U f)\lambda = \nabla_U^{\perp}(f\lambda) - f\nabla_U^{\perp}\lambda, \quad (4.4)$$

for any vector fields U and V tangent to M and any vector field λ normal to M [2].

DEFINITION 4.2: The endomorphism P (resp. the endomorphism f , the 1-forms F and t) is parallel if $\bar{\nabla}P = 0$ (resp. $\bar{\nabla}f = 0$, $\bar{\nabla}F = 0$ and $\bar{\nabla}t = 0$).

By virtue of (1.5) and (1.6), we can prove

PROPOSITION 4.1: For the covariant differentiations defined in DEFINITION 4.1, we have

$$(\bar{\nabla}_U P)V = \langle U, V \rangle \xi + \eta(V)U + t\sigma(U, V) + A_{FV}U, \quad (4.5)$$

$$(\bar{\nabla}_U F)V = f\sigma(U, V) - \sigma(U, PV), \quad (4.6)$$

$$(\bar{\nabla}_U t)\lambda = A_{f\lambda}U - PA_\lambda U, \quad (4.7)$$

$$(\nabla_U f)\lambda = -FA_\lambda U - \sigma(U, t\lambda) \quad (4.8)$$

for any vector fields U and V tangent to M and any vector field λ normal to M .

By virtue of (4.5), we get

$$(\bar{\nabla}_X P)Y = -\langle X, Y \rangle \xi + \eta(Y)X + t\sigma(X, Y) \quad (4.9)$$

for any X and Y in D . Thus we have

PROPOSITION 4.2: In a contact CR-submanifold M of a Sasakian manifold \tilde{M} , the structure $(P, \xi, \eta, \langle, \rangle)$ is Sasakian if and only if $\sigma(X, Y)$ is in Π for any X and Y in D .

COROLLARY 4.3: In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , if $\dim D_x^\perp = 0$, then the submanifold M is a Sasakian submanifold.

Next, we assume that the endomorphism P is parallel. Then we have from (4.9)

$$t\sigma(X, Y) = \langle X, Y \rangle \xi - \eta(Y)X \quad (4.10)$$

for any X and Y in D . By virtue of $\sigma(X, \xi) = 0$ and (4.10), we have $X = \alpha\xi$ for any X in D , where α is a certain scalar field on D . Thus we have

PROPOSITION 4.4: In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , if the endomorphism P is parallel, then $\dim D_x^\perp = 1$.

5. THE DISTRIBUTION D^\perp .

In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , we assume that the leaf M^\perp of D^\perp is totally geodesic in M , that is, $\nabla_Z W$ is in D^\perp for any Z and W in D^\perp . This means

$$\langle \nabla_Z W, \phi X \rangle = 0 \quad (5.1)$$

for any X in D and Z and W in D^\perp . By virtue of PROPOSITION 3.4 and (5.1), we have

$$\langle \sigma(X, Z) - \eta(X)\phi Z, \phi W \rangle = 0 \quad (5.2)$$

for any X in D and Z and W in D^\perp .

Conversely, if the equation (5.2) is satisfied, then it is clear that the leaf M^\perp of D^\perp is totally geodesic in M . Thus we have

PROPOSITION 5.1: In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , the leaf M^\perp of D^\perp is totally geodesic in M if and only if the equation (5.2) is satisfied.

Next, let us prove

THEOREM 5.2: In a contact CR -submanifold M of a Sasakian manifold \tilde{M} , we assume that the leaf M^\perp of D^\perp is totally geodesic in M . If the endomorphism P satisfies

$$(\tilde{\nabla}_U P)V = \eta(V)U - \langle U, V \rangle \xi \quad (5.3)$$

for any vector fields U and V tangent to M , then $\dim D_x^\perp = 0$, that is, the submanifold M is a Sasakian one.

PROOF: The equation (5.2) means

$$\langle A_{\phi W} X - \eta(X)W, Z \rangle = 0 \quad (5.4)$$

for any X in D and Z and W in D^\perp .

On the other hand, we have from (5.3)

$$t\sigma(U, V) + A_{FV}U = 0$$

for any vector fields U and V tangent to M . From this, we obtain $t\sigma(U, X) = 0$ for any vector field U tangent to M and X in D . Thus we have

$$\langle \sigma(U, X), \phi W \rangle = \langle A_{\phi W}X, U \rangle = -\langle \phi \sigma(U, X), W \rangle = 0,$$

that is, $A_{\phi W}X = 0$ for any X in D and W in D^\perp . Substituting this equation into (5.4), we get $\dim D_x^\perp = 0$.

6. A CONTACT CR-PRODUCT OF A SASAKIAN MANIFOLD I.

In this section, we shall define a contact CR-product and give a necessary and sufficient condition that a contact CR-submanifold is a contact CR-product.

DEFINITION 6.1: A contact CR-submanifold M of a Sasakian manifold \tilde{M} is called a contact CR-product if it is locally product of M^\perp and M^\parallel , where M^\parallel denotes the leaf of the distribution D .

THEOREM 6.1: A contact CR-submanifold M of a Sasakian manifold \tilde{M} is a contact CR-product if and only if

$$A_{\phi W}X = \eta(X)W \tag{6.1}$$

for any X in D and W in D^\perp .

PROOF: Since (6.1) means

$$\langle \sigma(X, Z) - \eta(X)\phi Z, \phi W \rangle = 0 \tag{6.2}$$

for any X in D and Z and W in D^\perp , the leaf M^\perp of D^\perp is totally geodesic in M . Furthermore, we have

$$\langle \sigma(X, \phi Y), \phi Z \rangle = \eta(X)\langle Z, \phi Y \rangle = 0$$

for any X and Y in D and Z in D^\perp . So, by virtue of PROPOSITION 3.4, the distribution D is integrable.

Let M^\parallel be the leaf of the distribution D , then we have from (6.1)

$$\begin{aligned} \langle \nabla_X Y, Z \rangle &= \langle \tilde{\nabla}_X Y, Z \rangle = \langle \phi \tilde{\nabla}_X Y, \phi Z \rangle = \langle \tilde{\nabla}_X \phi Y - (\tilde{\nabla}_X \phi)Y, \phi Z \rangle = \langle \sigma(X, \phi Y), \phi Z \rangle \\ &= \langle A_{\phi Z}X, Y \rangle = 0 \end{aligned}$$

for any X and Y in D and Z in D^\perp , that is, the leaf M^\parallel of D is totally geodesic in M . Thus the submanifold M is a contact CR-product.

Conversely, if the submanifold M of a Sasakian manifold \tilde{M} is a contact CR -product, then we have from (5.2)

$$A_{\phi W}X - \eta(X)W \in D \tag{6.3}$$

for any X in D and W in D^\perp . So, it is sufficient to prove the following:

$$A_{\phi W}X - \eta(X)W \in D^\perp \tag{6.4}$$

for any X in D and W in D^\perp . In fact, since the distribution D is totally geodesic in M , we have

$$\begin{aligned} \langle A_{\phi W}X - \eta(X)W, Y \rangle &= \langle \sigma(X, Y), \phi W \rangle = -\langle \phi \sigma(X, Y), W \rangle = -\langle \phi(\tilde{\nabla}_X Y - \nabla_X Y), W \rangle \\ &= -\langle \phi \tilde{\nabla}_X Y, W \rangle = \langle \nabla_X \phi Y, W \rangle = 0 \end{aligned}$$

for any X and Y in D and W in D^\perp . This means (6.4). By virtue of (6.3) and (6.4), we have (6.1).

7. A CONTACT CR -PRODUCT OF A SASAKIAN MANIFOLD II.

In this section, we shall mainly study the second fundamental form of a contact CR -product.

Let M be a contact CR -product of a Sasakian manifold \tilde{M} . In M , we shall calculate the $\tilde{H}_B(X, Z)$ for any unit vectors X in D and Z in D^\perp , where $\tilde{H}_B(X, Z)$ is defined by

$$\tilde{H}_B(X, Z) = -\langle \tilde{R}(X, \phi X)Z, \phi Z \rangle. \tag{7.1}$$

By virtue of (1.14) and (1.16), we get

$$\begin{aligned} \langle R(\tilde{X}, \phi X)Z, \phi Z \rangle &= \langle \nabla_X^\perp \sigma(\phi X, Z), \phi Z \rangle - \langle \nabla_{\phi X}^\perp \sigma(X, Z), \phi Z \rangle - \langle A_{\phi Z} \nabla_X X, Z \rangle \\ &- \langle A_{\phi Z} X, \nabla_X Z \rangle + \langle A_{\phi Z} \nabla_{\phi X} X, Z \rangle + \langle A_{\phi Z} X, \nabla_{\phi X} Z \rangle. \end{aligned}$$

Since the leaves M^\perp and M^1 are both totally geodesic in M , we have

$$\nabla_U Y \in D \quad \text{and} \quad \nabla_U Z \in D^\perp \tag{7.2}$$

for any vector field U tangent to M, Y in D and Z in D^\perp . Thus we have from (6.1) and (7.2)

$$\langle \tilde{R}(X, \phi X)Z, \phi Z \rangle = \langle \nabla_X^\perp \sigma(\phi X, Z), \phi Z \rangle - \langle \nabla_{\phi X}^\perp \sigma(X, Z), \phi Z \rangle - \eta(\nabla_X \phi X) + \eta(\nabla_{\phi X} X). \tag{7.3}$$

On the other hand, we obtain

$$\eta(\nabla_X \phi X) = \langle \nabla_X \phi X, \xi \rangle = \langle \tilde{\nabla}_X \phi X, \xi \rangle = -1 + \eta(X)^2.$$

So, (7.3) can be written as

$$\begin{aligned} \langle \tilde{R}(X, \phi X)Z, \phi Z \rangle &= \langle \nabla_X^\perp \sigma(\phi X, Z), \phi Z \rangle - \langle \nabla_{\phi X}^\perp \sigma(X, Z), \phi Z \rangle + 1 - \eta(X)^2 \\ &+ \eta(\nabla_{\phi X} X). \end{aligned} \quad (7.4)$$

Next, we have from (6.1)

$$\langle \sigma(X, W), \phi Z \rangle = \langle \xi, X \rangle \langle Z, W \rangle, \quad (7.5)$$

from which

$$\langle \sigma(X, Z), \phi Z \rangle = \langle \xi, X \rangle, \quad (7.6)$$

$$\langle \sigma(\phi X, Z), \phi Z \rangle = 0. \quad (7.7)$$

Covariant differentiation of (7.6) and (7.7) along ϕX and X respectively give us

$$\langle \nabla_{\phi X}^\perp \sigma(X, Z), \phi Z \rangle = -\langle \sigma(X, Z), \nabla_{\phi X}^\perp \phi Z \rangle + \langle \nabla_{\phi X} \xi, X \rangle + \eta(\nabla_{\phi X} X), \quad (7.8)$$

$$\langle \nabla_X^\perp \sigma(\phi X, Z), \phi Z \rangle = -\langle \sigma(\phi X, Z), \nabla_X^\perp \phi Z \rangle. \quad (7.9)$$

Substituting (7.8) and (7.9) into (7.4), we get

$$\begin{aligned} \langle \tilde{R}(X, \phi X)Z, \phi Z \rangle &= -\langle \sigma(X, Z), \nabla_{\phi X}^\perp \phi Z \rangle + \langle \sigma(\phi X, Z), \nabla_X^\perp \phi Z \rangle \\ &+ 2(1 - \eta(X)^2). \end{aligned} \quad (7.10)$$

By virtue of (3.9) and (6.1), we can calculate

$$\langle \sigma(\phi X, Z), \nabla_X^\perp \phi Z \rangle - \langle \sigma(X, Z), \nabla_{\phi X}^\perp \phi Z \rangle = -2\|\sigma(X, Z)\|^2 + 2\eta(X)^2. \quad (7.11)$$

Thus we have

PROPOSITION 7.1: In a contact CR-product of a Sasakian manifold, we have

$$\tilde{H}_B(X, Z) = 2(\|\sigma(X, Z)\|^2 - 1) \quad (7.12)$$

for any unit vectors X in D and Z in D^\perp .

Especially, if the ambient manifold \tilde{M} is a Sasakian space form $\tilde{M}(k)$, then we have from (1.15)

$$\tilde{R}(X, \phi X; \phi Z, Z) = \frac{k-1}{2} (1 - \eta(X)^2). \quad (7.13)$$

Thus we have

PROPOSITION 7.2: In a contact CR-product of a Sasakian space form $\tilde{M}(k)$, we have

$$\|\sigma(X, Z)\|^2 = \frac{k+3}{4} - \frac{k-1}{4} \eta(X)^2 \tag{7.14}$$

for any unit vectors X in D and Z in D^\perp .

By virtue of PROPOSITION 3.4. and (7.14), we have

$$\left. \begin{aligned} \|\sigma(X, Z)\|^2 &= \frac{k+3}{4} && \text{for } X \perp \xi, \\ \|\sigma(\xi, Z)\|^2 &= 1 \end{aligned} \right\} \tag{7.15}$$

Thus we have

COROLLARY 7.3: In a Sasakian space form $\tilde{M}(k)$ with constant ϕ -holomorphic sectional curvature $k < -3$, there does not exist a contact CR-product of $\tilde{M}(k)$.

Next, we shall prove

THEOREM 7.4: Let M be a contact CR-submanifold of a Sasakian space form $\tilde{M}(k)$.

Then we have

$$\|\sigma\|^2 \geq 2p \left(\frac{h(k+3)}{2} + 1 \right), \tag{7.16}$$

where $p = \dim D^\perp$ and $2h = \dim D - 1$. If the equality sign of (7.16) holds, then M^\perp and M^1 are both totally geodesic in $\tilde{M}(k)$.

PROOF: Let $A_1, A_2, \dots, A_h, \phi A_1, \phi A_2, \dots, \phi A_h, A_{2h+1}$ ($= \xi$) and B_1, B_2, \dots, B_p be orthogonal basis of D_x and D_x^\perp , respectively. Then $\|\sigma\|^2$ is given by

$$\|\sigma\|^2 = \sum_{i,j=1}^{2h+1} \|\sigma(A_i, A_j)\|^2 + 2 \sum_{i=1}^{2h+1} \sum_{\alpha=1}^p \|\sigma(A_i, B_\alpha)\|^2 + \sum_{\alpha, \beta=1}^p \|\sigma(B_\alpha, B_\beta)\|^2.$$

By virtue of (7.15), the above equation can be written as

$$\|\sigma\|^2 = 2p \left(\frac{h(k+3)}{2} + 1 \right) + \sum_{i,j=1}^{2h+1} \|\sigma(A_i, A_j)\|^2 + \sum_{\alpha, \beta=1}^p \|\sigma(B_\alpha, B_\beta)\|^2. \tag{7.17}$$

From the above equation, we have our theorem.

8. TOTALLY UMBILICAL CONTACT CR-SUBMANIFOLDS.

Let M be a totally umbilical contact CR-submanifold of a Sasakian manifold \tilde{M} .

Then by definition we have

$$\sigma(U, V) = \langle U, V \rangle H \tag{8.1}$$

for any vector fields U and V tangent to M . By virtue of (1.6), we can write

$$\phi H = tH + fH. \tag{8.2}$$

Since the vector field tH is in D^\perp , we have from (3.8)

$$A_{\phi tH}W = A_{\phi W}tH \quad (8.3)$$

for any W in D^\perp . From this, we obtain

$$-\langle W, W \rangle \langle tH, tH \rangle = \langle tH, W \rangle \langle H, \phi W \rangle. \quad (8.4)$$

We assume that $\dim D^\perp \geq 2$. Then we can put W as the orthogonal vector field of tH .

The equation (8.4) means

$$tH = 0 \quad (8.5)$$

Next, let Q_1 and Q_2 be the projections of TM to D and D^\perp , respectively. Then for any vector field U tangent to M we can put

$$U = Q_1U + Q_2U \quad (8.6)$$

and

$$\phi Q_1U \in D, \quad \phi Q_2U \in \phi D^\perp \subset T^\perp M. \quad (8.7)$$

The equation (8.7) and the covariant differentiation of $\phi\lambda = t\lambda + f\lambda$ teach us

$$-\phi Q_1 A_\lambda U = Q_1 \nabla_U t\lambda - Q_1 A_{f\lambda} U \quad (8.8)$$

for any vector field U tangent to M and any vector field λ normal to M . In (8.8), if we put $\lambda = H$ and taking account of (8.5), we have

$$\phi Q_1 A_H U = Q_1 A_{\phi H} U \quad (8.9)$$

for any vector field U tangent to M . For any X in D and any vector field U tangent to M , we get

$$\langle Q_1 A_{\phi H} U, X \rangle = \langle A_{\phi H} U, X \rangle = \langle \sigma(U, X), \phi H \rangle = \langle U, X \rangle \langle H, \phi H \rangle = 0,$$

$$\langle \phi Q_1 A_H U, X \rangle = -\langle Q_1 A_H U, \phi X \rangle = -\langle A_H U, \phi X \rangle = -\langle \sigma(U, \phi X), H \rangle = -\langle U, \phi X \rangle \langle H, H \rangle.$$

By virtue of (8.9) and the above two equations, we have

$$\langle H, H \rangle \langle U, \phi X \rangle = 0 \quad (8.10)$$

for any vector field U tangent to M and any X in D . We assume that $\dim D_x \geq 2$ and if we take $U = \phi X$ such that the vector field X is orthogonal to ξ , we have from (8.10) $H = 0$. Thus we have

THEOREM 8.1: Let M be a totally umbilical contact CR-submanifold of a Sasakian

manifold \tilde{M} . We assume that $\dim D_x \geq 2$ and $\dim D_x^\perp \geq 2$. Then the submanifold M is totally geodesic in \tilde{M} .

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