

## ON THE RADIUS OF UNIVALENCE OF CONVEX COMBINATIONS OF ANALYTIC FUNCTIONS

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ABSTRACT. We consider for  $\alpha > 0$ , the convex combinations  $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$ , where  $F$  belongs to different subclasses of univalent functions and find the radius for which  $f$  is in the same class.

KEY WORDS AND PHRASES. *Univalent functions, alpha-quasi-convex, starlike, close-to-convex functions, convex combinations.*

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### 1. INTRODUCTION.

Let  $S$ ,  $K$ ,  $S^*$  and  $C$  denote the classes of analytic functions in the unit disc  $E = \{z: |z| < 1\}$  which are respectively univalent, close-to-convex, starlike, and convex. In [1,2], a new subclass  $C^*$  of univalent functions was introduced and studied. A function  $f$ , analytic in  $E$ , belongs to  $C^*$  if and only if there exists a convex function  $g$  such that for  $z \in E$ ,

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0. \quad (1.1)$$

The functions in  $C^*$  are called quasi-convex and  $C \subset C^* \subset K \subset S$ . It is shown [2] that  $f \in C^*$  if and only if  $zf' \in K$ . Recently the functions called  $\alpha$ -quasi-convex have been defined and their properties studied in [3]. A function  $f$ , analytic in  $E$ , is said to be  $\alpha$ -quasi-convex if and only if there exists a convex function  $g$  such that, for  $\alpha$  real and positive

$$\operatorname{Re} \left\{ (1 - \alpha) \frac{f'(z)}{g'(z)} + \alpha \frac{(zf'(z))'}{g'(z)} \right\} > 0. \quad (1.2)$$

It has been shown [3] that  $F$  is  $\alpha$ -quasi-convex if and only if  $f$  with

$$f(z) = (1 - \alpha)F(z) + \alpha zF'(z) \text{ is close-to-convex.} \quad (1.3)$$

All  $\alpha$ -quasi-convex functions are close-to-convex.

## 2. MAIN RESULTS.

We shall now study the mapping properties of  $f$ :  $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$ ,  $\alpha > 0$ , when  $F$  belongs to different subclasses of univalent functions.

**THEOREM 2.1.** Let  $F \in S^*$  and  $\alpha > 0$ . The function

$$F(z) = (1 - \alpha)F(z) + \alpha zF'(z) \quad (2.1)$$

is starlike in  $|z| < r_0$ , where

$$r_0 = \frac{1}{2\alpha + \sqrt{4\alpha^2 + 1 - 2\alpha}}. \quad (2.2)$$

This result is sharp.

**PROOF.** We can write (2.1) as

$$f(z) = \alpha z^{2 - \frac{1}{\alpha}} (z^{\frac{1}{\alpha} - 1} F(z))',$$

and from this it follows that

$$F(z) = \frac{1}{\alpha} z^{1 - \frac{1}{\alpha}} \int_0^z z^{\frac{1}{\alpha} - 2} f(z) dz. \quad (2.3)$$

Then

$$\begin{aligned} \frac{zF'(z)}{F(z)} &= \left\{ \left(1 - \frac{1}{\alpha}\right) z^{1 - \frac{1}{\alpha}} \int_0^z z^{\frac{1}{\alpha} - 2} f(z) dz + f(z) \right\} / \left\{ z^{1 - \frac{1}{\alpha}} \int_0^z z^{\frac{1}{\alpha} - 2} f(z) dz \right\} \\ &= \left\{ \left(1 - \frac{1}{\alpha}\right) \int_0^z z^{\frac{1}{\alpha} - 2} f(z) dz + z^{\frac{1}{\alpha} - 1} f(z) \right\} / \left\{ \int_0^z z^{\frac{1}{\alpha} - 2} f(z) dz \right\} = h(z), \end{aligned} \quad (2.4)$$

where  $\operatorname{Re} h(z) > 0$ , since  $F \in S^*$ .

From (2.4), we have

$$z^{\frac{1}{\alpha} - 1} f(z) - \left(\frac{1}{\alpha} - 1\right) \int_0^z z^{\frac{1}{\alpha} - 2} f(z) dz = h(z) \int_0^z z^{\frac{1}{\alpha} - 2} f(z) dz. \quad (2.5)$$

Differentiating both sides of (2.5), we obtain

$$\left(\frac{1}{\alpha} - 1\right) z^{\frac{1}{\alpha} - 2} f(z) + z^{\frac{1}{\alpha} - 1} f'(z) - \left(\frac{1}{\alpha} - 1\right) z^{\frac{1}{\alpha} - 2} f(z) = h'(z) \int_0^z z^{\frac{1}{\alpha} - 2} f(z) dz + h(z) z^{\frac{1}{\alpha} - 2} f(z).$$

Thus

$$\frac{zf'(z)}{f(z)} = h(z) + \{h'(z) \int_0^z z^{\frac{1}{\alpha} - 2} f(z) dz\} / \{z^{\frac{1}{\alpha} - 2} f(z)\}.$$

Now, using the well-known result [4],  $|h'(z)| \leq \{2\text{Re } h(z)\}/(1 - r^2)$ ,  $|z| = r$ , we have

$$\text{Re } \frac{zf'(z)}{f(z)} \geq \text{Re } h(z) \left\{ 1 - \frac{2}{1 - r^2} \left| \frac{\int_0^z \frac{z^{\frac{1}{\alpha}-2} f(z) dz}{z^{\frac{1}{\alpha}-2} f(z)} \right| \right\}. \tag{2.6}$$

From (2.1) and (2.3), we have

$$\begin{aligned} \frac{z^{\frac{1}{\alpha}-1} f(z)}{\int_0^z z^{\frac{1}{\alpha}-2} f(z) dz} &= \frac{\alpha z(z^{\frac{1}{\alpha}-1} F(z))'}{\alpha(z^{\frac{1}{\alpha}-1} F(z))} = \frac{z\{z^{\frac{1}{\alpha}-1} F'(z) + (\frac{1}{\alpha} - 1)z^{\frac{1}{\alpha}-2} F(z)\}}{(z^{\frac{1}{\alpha}-1} F(z))} \\ &= \frac{zF'(z)}{F(z)} + (\frac{1}{\alpha} - 1) = h(z) + (\frac{1}{\alpha} - 1), \end{aligned}$$

from which it follows that

$$\left| \{z^{\frac{1}{\alpha}-1} f(z) / \int_0^z z^{\frac{1}{\alpha}-2} f(z) dz\} \right| \geq \text{Re}\{h(z) + (\frac{1}{\alpha} - 1)\} \geq (\frac{1}{\alpha} - 1) + \frac{1 - r}{1 + r}. \tag{2.7}$$

Using (2.7), we have from (2.6)

$$\begin{aligned} \text{Re } \frac{zf'(z)}{f(z)} &\geq \text{Re } h(z) \left\{ 1 - \left( \frac{2}{1 - r^2} \right) \left( \frac{r + r^2}{\frac{1}{\alpha} + (\frac{1}{\alpha} - 2)r} \right) \right\} \\ &= \text{Re } h(z) \left\{ \left( \frac{1}{\alpha} - 4r - (\frac{1}{\alpha} - 2)r^2 \right) / \left\{ (1 - r) \left( \frac{1}{\alpha} + (\frac{1}{\alpha} - 2)r \right) \right\} \right\}. \end{aligned} \tag{2.8}$$

The right hand side of (2.8) is positive for  $r < r_0$ , where  $r_0$  is given by (2.2). This result is sharp as can be seen by

$$\begin{aligned} f_0(z) &= \{\alpha(z(\frac{1}{\alpha} - (\frac{1}{\alpha} - 2)z))\}/(1 - z)^3 \\ &= (1 - \alpha)F_0(z) + \alpha z F_0'(z), \end{aligned} \tag{2.9}$$

where

$$F_0(z) = \frac{z}{(1 - z)^2} \in S^*,$$

REMARK 2.1. Let  $f \in C$ , then  $f$ , given by (2.1), is convex for  $|z| < r_0$ , where  $r_0$  is given by (2.2). The proof follows on the same lines as in Theorem 2.1. See also [5] and [6].

REMARK 2.2. In [6], Nikolaeva and Repnina treated the same problem, with a different notation, for the convex and starlike functions of order  $\beta$ . Theorem 2.1 follows from their result when we take  $\beta = 0$  for  $0 \leq \alpha \leq 1$ . On the other hand, our proof of Theorem 2.1 is much simpler and the result holds for all  $\alpha > 0$ .

**THEOREM 2.2.** Let  $F \in K$  and  $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$ ,  $\alpha > 0$ . Then  $f$  is close-to-convex in  $|z| < r_0$ ,  $r_0$  is given by (2.2). The function  $f_0$  in (2.9) shows that this result is sharp.

**PROOF.** Since  $F \in K$ , there exists a  $G \in S^*$  such that, for  $z \in E$ ,  $\text{Re} \frac{zF'(z)}{G(z)} > 0$ . Now let  $g(z) = (1 - \alpha)G(z) + \alpha zG'(z)$ . Then by Theorem 2.1,  $g$  is starlike for  $|z| < r_0$ ,  $r_0$  is defined by (2.2). Using the same technique of Theorem 2.1, we can easily show that  $\text{Re} \frac{zf'(z)}{g(z)} > 0$  for  $|z| < r_0$ .

**REMARK 2.3.** For  $\alpha = \frac{1}{2}$ , this result has been proved in [7].

As an easy consequence of (1.3) and Theorem 2.2, we have the following.

**COROLLARY 2.1.** Let  $F \in K$  and  $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$ ,  $\alpha > 0$ . Then  $F$  is  $\alpha$ -quasi-convex in  $|z| < r_0$ . This means that the radius of  $\alpha$ -quasi-convexity for close-to-convex functions is given by (2.2).

**THEOREM 2.3.** Let  $F \in C^*$  and  $\alpha > 0$ . Let  $f(z) = (1 - \alpha)F(z) + \alpha zF'(z)$ . Then  $f$  is in  $C^*$ , for  $|z| < r_0$ ,  $r_0$  is given by (2.2).

**PROOF.** Since  $F \in C^*$ , there exists a  $G \in C$  such that for  $z \in E$ ,  $\text{Re} \frac{(zF'(z))'}{G'(z)} > 0$ . Now let  $g(z) = (1 - \alpha)G(z) + \alpha zG'(z)$ , then  $g$  is convex in  $|z| < r_0$ . We can write

$$f(z) = (1 - \alpha)F(z) + \alpha zF'(z) = z^{2 - \frac{1}{\alpha}} \left( z^{\frac{1}{\alpha} - 1} F(z) \right)'$$

and

$$g(z) = (1 - \alpha)G(z) + \alpha zG'(z) = z^{2 - \frac{1}{\alpha}} \left( z^{\frac{1}{\alpha} - 1} G(z) \right)'$$

Thus

$$\frac{(zf'(z))'}{g'(z)} = \left( \left( z \left( z^{2 - \frac{1}{\alpha}} \left( z^{\frac{1}{\alpha} - 1} F(z) \right)' \right)' \right)' \right)' / \left( z \left( z^{2 - \frac{1}{\alpha}} \left( z^{\frac{1}{\alpha} - 1} G(z) \right)' \right)' \right)' \tag{2.10}$$

Now

$$\begin{aligned} \left( z \left( z^{2 - \frac{1}{\alpha}} \left( z^{\frac{1}{\alpha} - 1} F(z) \right)' \right)' \right)' &= \left( z \left( \left( \frac{1}{\alpha} - 1 \right) F(z) + zF'(z) \right)' \right)' = \left( \frac{1}{\alpha} zF'(z) + z^2 F''(z) \right)' \\ &= \left( z^{2 - \frac{1}{\alpha}} \left( \frac{1}{\alpha} z^{\frac{1}{\alpha} - 1} F'(z) + z^{\frac{1}{\alpha}} F''(z) \right) \right)' = \left( z^{2 - \frac{1}{\alpha}} \left( z^{\frac{1}{\alpha} - 1} zF'(z) \right)' \right)' \end{aligned}$$

Let  $zF'(z) = H(z)$ , then from (2.10), we have

$$\frac{(zf'(z))'}{g'(z)} = \left( z^{2 - \frac{1}{\alpha}} \left( z^{\frac{1}{\alpha} - 1} H(z) \right)' \right)' / \left( z^{2 - \frac{1}{\alpha}} \left( z^{\frac{1}{\alpha} - 1} G(z) \right)' \right)'$$

Since from Theorem 2.2, the function  $(1 - \alpha)H(z) + zH'(z) = z^{2 - \frac{1}{\alpha}} \left( z^{\frac{1}{\alpha} - 1} H(z) \right)'$  belongs to  $K$  with respect to a convex function  $g$ :  $g(z) = (1 - \alpha)G(z) + \alpha zG'(z)$  in

$|z| < r_0$ , so  $f$  is in  $C^*$  for  $|z| < r_0$ , where  $r_0$  is given by (2.2).

REMARK 2.4. For  $F \in C^*$  and  $\alpha = \frac{1}{2}$ , Theorem 2.3 has been proved in [1].

We now deal with a generalized form of (1.1) by taking  $g$  to be starlike and prove the following.

THEOREM 2.4. Let  $F$  be analytic in  $E$  and let for  $z \in E$ ,  $\text{Re} \frac{(zF'(z))'}{G'(z)} > 0$ ,  $G \in S^*$ .

Let  $f(z) = (1-\alpha)F(z) + \alpha zF'(z)$  and  $g(z) = (1-\alpha)G(z) + \alpha zG'(z)$ , with  $\alpha > 0$ . Then

$\text{Re} \frac{(zf'(z))'}{g'(z)} > 0$  for  $|z| < r_1$ , where

$$r_1 = \frac{1}{3\alpha + \sqrt{9\alpha^2 + 1 - 2\alpha}}$$

For  $\alpha = \frac{1}{2}$ , the problem has been solved in [8].

PROOF. From (2.3), we can write

$$F(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z z^{\frac{1}{\alpha}-2} f(z) dz$$

$$\begin{aligned} zF'(z) &= \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \left( (1-\frac{1}{\alpha}) \int_0^z z^{\frac{1}{\alpha}-2} f(z) dz + z^{\frac{1}{\alpha}-1} f(z) \right) \\ &= \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \left( \int_0^z z^{\frac{1}{\alpha}-1} f'(z) dz \right). \end{aligned}$$

Thus

$$\frac{(zF'(z))'}{G'(z)} = \frac{\frac{1}{\alpha} z^{\frac{1}{\alpha}} f'(z) - \left(\frac{1}{\alpha}-1\right) \int_0^z z^{\frac{1}{\alpha}-1} f'(z) dz}{\int_0^z z^{\frac{1}{\alpha}-1} g'(z) dz} = h(z), \tag{2.11}$$

where  $\text{Re} h(z) > 0$ ,  $z \in E$ .

From (2.11), we write

$$z^{\frac{1}{\alpha}} f'(z) - \left(\frac{1}{\alpha}-1\right) \int_0^z z^{\frac{1}{\alpha}-1} f'(z) dz = h(z) \int_0^z z^{\frac{1}{\alpha}-1} g'(z) dz.$$

Differentiating both sides, and simplifying, we obtain

$$\frac{(zf'(z))'}{g'(z)} = h(z) + \frac{h'(z) \left( \int_0^z z^{\frac{1}{\alpha}-1} g'(z) dz \right)}{z^{\frac{1}{\alpha}-1} g'(z)}. \tag{2.12}$$

Using  $|h'(z)| \leq \frac{2\text{Re} h(z)}{1-r^2}$ , (2.12) gives

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} \geq \operatorname{Re} h(z) \left[ 1 - \frac{2}{1-r^2} \left| \left( \int_0^z z^{\frac{1}{\alpha}-1} g'(z) dz \right) / \left( z^{\frac{1}{\alpha}-1} g'(z) dz \right) \right| \right]. \quad (2.13)$$

Now

$$\frac{\frac{1}{\alpha} g'(z)}{\left( \int_0^z z^{\frac{1}{\alpha}-1} g'(z) dz \right)} = \frac{(1/\alpha)G'(z) + zG''(z)}{G'(z)} = \left( \frac{1}{\alpha} - 1 \right) + \frac{(zG'(z))'}{G'(z)}. \quad (2.14)$$

Since  $G \in S^*$ , so

$$\left| \frac{(zG'(z))'}{G'(z)} \right| \geq \frac{1-4r+r^2}{1-r^2}. \quad (2.15)$$

From (2.13), (2.14) and (2.15), we obtain

$$\begin{aligned} \operatorname{Re} \frac{(zf'(z))'}{g'(z)} &\geq \operatorname{Re} h(z) \left[ 1 - \frac{2}{1-r^2} \frac{r(1-r^2)}{\frac{1}{\alpha} - 4r - \left( \frac{1}{\alpha} - 2 \right) r^2} \right] \\ &= \operatorname{Re} h(z) \frac{1-6\alpha r - (1-2\alpha)r^2}{1-4\alpha r - (1-2\alpha)r^2}, \end{aligned}$$

and this positive for  $|z| < r_1$ , where

$$r_1 = \frac{1}{3\alpha + \sqrt{9\alpha^2 + 1 - 2\alpha}}.$$

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