

## FUNCTIONS IN THE SPACE $R^2(E)$ AT BOUNDARY POINTS OF THE INTERIOR

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**ABSTRACT.** Let  $E$  be a compact subset of the complex plane  $\mathbb{C}$ . We denote by  $R(E)$  the algebra consisting of (the restrictions to  $E$  of) rational functions with poles off  $E$ . Let  $m$  denote 2 - dimensional Lebesgue measure. For  $p \geq 1$ , let  $R^p(E)$  be the closure of  $R(E)$  in  $L^p(E, dm)$ .

In this paper we consider the case  $p = 2$ . Let  $x \in \partial E$  be a bounded point evaluation for  $R^2(E)$ . Suppose there is a  $C > 0$  such that  $x$  is a limit point of the set  $S = \{y \mid y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y - x|\}$ . For those  $y \in S$  sufficiently near  $x$  we prove statements about  $|f(y) - f(x)|$  for all  $f \in R(E)$ .

**KEY WORDS AND PHRASES.** Rational functions, compact set  $L^p$  - spaces, bounded point evaluation, admissible function.

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### 1. INTRODUCTION AND DEFINITIONS

Let  $E$  be a compact subset of the complex plane  $\mathbb{C}$ . We denote by  $R(E)$  the algebra consisting of (the restrictions to  $E$  of) rational functions with poles off  $E$ . Let  $m$  denote 2 - dimensional Lebesgue measure. For  $p \geq 1$ , let  $R^p(E)$  be the closure of  $R(E)$  in  $L^p(E, dm)$ . A point  $x \in E$  is said to be a bounded point evaluation (BPE) for  $R^p(E)$  if there is a constant  $F$  such that

$$|f(x)| \leq F \cdot \left[ \int_E |f(z)|^p dm(z) \right]^{\frac{1}{p}} \text{ for all } f \in R(E).$$

In [4] we studied the smoothness properties of functions in  $R^p(E)$ ,  $p > 2$ , at BPE's. When  $p = 2$ , the situation is quite different (see Fernström and Polking [2] and Fernström [1]). In [5] we showed that at certain BPE's the functions in  $R^2(E)$  have the following smoothness property: Let  $x \in \partial E$  be both a BPE for  $R^2(E)$  and the vertex of a sector contained in  $\text{Int } E$ . Let  $L$  be a line segment that bisects the sector and has an end point at  $x$ . Then for each  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $y \in L$  and  $|y - x| < \delta$ ,  $|f(y) - f(x)| \leq \epsilon \|f\|_2$  for all  $f \in R(E)$ . The goal of this paper is to extend this result to certain cases where there may not be a sector in  $\text{Int } E$  having vertex at  $x$ , but  $x$  is still a limit point of  $\text{Int } E$ .

If  $x \in E$  is a BPE for  $R^2(E)$ , there is a function  $g \in L^2(E)$  such that  $f(x) = \int_E fg \, dm$  for any  $f \in R(E)$ . Such a function  $g$  is called a representing function for  $x$ .

A point  $x \in E$  is a bounded point derivation (BPD) of order  $s$  for  $R^2(E)$  if the map  $f \rightarrow f^{(s)}(x)$ ,  $f \in R(E)$ , extends from  $R(E)$  to a bounded linear functional on  $R^2(E)$ .

Let  $A_n(x)$  denote the annulus  $\{z | 2^{-n-1} \leq |z - x| \leq 2^{-n}\}$ . Let  $A'_n(x) = \{z | 2^{-n-2} \leq |z - x| \leq 2^{-n+1}\}$ . If  $x = 0$ , we will denote  $A_n(0)$  by  $A_n$  and  $A'_n(0)$  by  $A'_n$ .

For an arbitrary set  $X \subset \mathbb{C}$  we let  $C_2(X)$  denote the Bessel capacity of  $X$  which is defined using the Bessel kernel of order 1 (see [3]).

We say that  $\phi$  is an admissible function if  $\phi$  is a positive, non-decreasing function defined on  $(0, \infty)$ , and  $r \cdot \phi(r)^{-1}$  is nondecreasing and tends to zero when  $r \rightarrow 0^+$ .

Using the techniques of [4] and [2] one can prove:

**THEOREM 1.1.** Let  $s$  be a nonnegative integer and  $E$  a compact set. Suppose that  $x$  is a BPE for  $R^2(E)$  and  $\phi$  is admissible. Then  $x$  is

represented by a function  $g \in L^2(E)$  such that

$$\frac{g}{(z-x)^s \cdot \phi(|z-x|)} \in L^2(E)$$

if and only if  $\sum_{n=0}^{\infty} 2^{2n(s+1)} \phi(2^{-n})^{-2} C_2(A_n(x)-E) < \infty$ .

2. THE MAIN RESULTS

Let  $x \in \partial E$  be a BPE for  $R^2(E)$ . We may assume that  $x = 0$  and that  $E \subset \{|z| < 1\}$ . Suppose there is a positive constant  $C$  such that  $0$  is a limit point of the set  $S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$ . We will construct a function  $g \in L^2(E)$  which represents  $0$  for  $R^2(E)$  and has support disjoint from  $S$ .

LEMMA 2.1. Let  $0 \in \partial E$  be a BPE for  $R^2(E)$ . Suppose there is a positive constant  $C$  such that  $0$  is a limit point of the set  $S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$ . Then there is a function  $g \in L^2(E)$  such that:

(i)  $g$  represents  $0$  for  $R^2(E)$ ,

(ii)  $m((\text{supp } g) \cap S) = 0$ ,

$$k = \left[ \frac{n-2}{2} \right] + 1$$

(iii) For all  $n \geq 2$ ,  $\int_{A_n \cap E} |g|^2 dm \leq F \sum 2^{2k} C_2(A'_{2k+1} - E)$

$$k = \left[ \frac{n-2}{2} \right]$$

where  $F$  is a constant independent of  $n$ .

PROOF. For each  $i, i = 0, 1, 2, \dots$  consider all the intersections of the set  $A_i = \{z | 2^{-i-1} \leq |z| \leq 2^{-i}\}$  with the bounded components of  $C - E$ . Let  $Y_i$  be the closure of the union of these intersections. Since  $Y_i$  is compact, it can be covered by finitely many open discs of radius  $< C3^{-1}2^{-i-1}$ . Let the union (finite) of these discs be denoted by  $B_i$ . The set  $B_i$  is bounded by finitely many closed Jordan curves each of which is the union of finitely many circular arcs. Each set  $B_i$  is contained in a set  $C_i$  bounded by finitely many closed Jordan curves  $\Gamma_{ij}, j = 1, 2, \dots, n_i$  such that if  $z$  belongs to any one of these

curves,  $\text{Dist}(z, B_i) = 2C3^{-1}2^{-i-1}$ .

Now for each  $k, k = 0, 1, 2, \dots$  choose a function  $\lambda_k \in C_0^1$

such that:

$$(1) \quad \text{supp } \lambda_k \subset A_{2k+1}'$$

$$(2) \quad \lambda_k(z) = 1 \text{ for } z \in \{|z|2^{-2k-2} \leq |z| \leq 2^{-2k-1}\} \cap B_{2k+1}$$

$$(3) \quad \lambda_k(z) = 0 \text{ for } z \notin \bigcup_{i=0}^{\infty} C_i$$

$$(4) \quad \left| \frac{\partial \lambda_k(z)}{\partial x_1} \right| \leq F_1 \cdot 2^{2k+1}, \quad \left| \frac{\partial \lambda_k(z)}{\partial x_2} \right| \leq F_2 \cdot 2^{2k+1}$$

where  $z = x_1 + ix_2$  and  $F_1$  and  $F_2$  are constants independent of  $k$ .

$$(5) \quad \lambda_k(z) + \lambda_{k+1}(z) = 1 \text{ for } z \in \{|z|2^{-2k-3} \leq |z| \leq 2^{-2k-2}\} \cap B_{2k+2}.$$

Given any  $\epsilon > 0$  we use a lemma of Fernström and Polking [2] to obtain functions  $\psi_k \in C^\infty$  such that:

$$(1) \quad \psi_k(z) \equiv 1 \text{ for } z \text{ near } A_k' - \{z \mid \text{Dist}(z, E) < \epsilon\}.$$

$$(2) \quad \int_{|z| \leq 2^{-k+1}} |D^\beta \psi_k(z)|^2 dm(z) \leq F \cdot 2^{-2k(1-|\beta|)} C_2(A_k' - E)$$

for  $\beta = (0, 0), (0, 1),$  and  $(1, 0)$ . Here the constant  $F$  is independent of  $\epsilon$  and  $k$ .

Since  $\text{supp } \lambda_k \subset A_{2k+1}'$ , we have  $\psi_{2k+1} \cdot \lambda_k = \lambda_k$  on the set  $\{z \mid \text{Dist}(z, E) \geq \epsilon\}$ . Thus  $\sum_0^\infty \psi_{2k+1} \cdot \lambda_k \equiv 1$  on  $\{|z| \leq 4^{-1}\} - \{z \mid \text{Dist}(z, E) < \epsilon\}$ . Choose  $\chi \in C_0^\infty$  with  $\chi(z) \equiv 1$  near  $E$ . Set  $h(z) = \chi(z) \cdot \frac{1}{\pi z}$ . For each double index  $\beta = (0, 0), (0, 1),$  and  $(1, 0)$  there is a constant  $F_\beta$  such that

$$|D^\beta h(z)| \leq F_\beta \cdot |z|^{-1-|\beta|}.$$

$$\text{Set } f_\epsilon = h \cdot \sum_0^\infty \psi_{2k+1} \cdot \lambda_k = \sum_0^\infty \psi_{2k+1} \cdot h_k$$

where  $h_k = \lambda_k \cdot h$ .

Since  $\text{supp } \lambda_k \subset A_{2k+1}$ , the above inequalities imply that

$$|D^\beta h_k(z)| \leq F_\beta 2^{(2k+1)(1+|\beta|)}$$

The subadditivity of  $C_2$  and the convergence of  $\sum_0^\infty 2^{2k} C_2(A_k - E)$  (see Theorem 1.1) imply that the net  $\{f_\epsilon\}$  is bounded in  $L^2_1$ . There is a subsequence that converges weakly to a function  $f \in L^2_{1,loc}$

which satisfies  $f(z) = \frac{\chi(z)}{\pi z}$  for  $z \in \mathbb{C} - E$  and  $f(z) = 0$  for every  $z \in E \cap \{z | \text{Dist}(z, \partial E) \geq C|z|\}$ . Set  $g = -\frac{\partial}{\partial \bar{z}} f$ . Then  $g \in L^2(E)$  since  $f \in L^2_1(E)$ , and  $g$  is a representing function for 0. The proof of (iii) proceeds as in [5].

The above lemma can be used to prove the following theorem in almost the same way that in [5] Lemma 5.1 is used to prove Theorem 5.1.

**THEOREM 2.1.** Let  $0 \in \partial E$  be a BPE for  $R^2(E)$ . Let  $C$  be a positive constant such that 0 is a limit point of the set  $S = \{y | y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C|y|\}$ . Let  $g$  be a representing function for 0 and suppose that  $g(z) \cdot \phi(|z|)^{-1} \in L^2(E)$  where  $\phi$  is an admissible function. Then for any  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $y \in S$  and  $|y| < \delta$ ,

$$|f(y) - f(0)| \leq \epsilon \phi(|y|) \|f\|_2$$

for all  $f \in R(E)$ .

Using this theorem and the methods in [4] one can prove:

**COROLLARY 2.1.** Suppose that all the conditions of Theorem 2.1 hold. Suppose, moreover, that  $s$  is a positive integer such that  $g(z) \cdot z^{-s} \cdot \phi(|z|)^{-1} \in L^2(E)$ . Then for each  $\epsilon > 0$  there exists a  $\delta > 0$  such that if  $y \in S$  and  $|y| < \delta$ ,

$$|f(y) - f(0) - \frac{f'(0)}{1!} (y - 0) - \dots - \frac{f^{(s)}(0)}{s!} (y - 0)^s| \leq \epsilon |y - 0|^s \phi(|y|) \|f\|_2$$

for all  $f \in R(E)$ .

Finally, there is a corollary with weaker preconditions.

COROLLARY 2.2. Let  $0, g$ , and  $\phi$  be as in Theorem 2.1. Suppose there is a positive constant  $C$  such that  $0$  is a limit point of the set

$$S = \{y \mid y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C \cdot \phi(|y|) |y|\}$$

Then for each  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $y \in S$  and  $|y| < \delta$ ,

$$|f(y) - f(0)| \leq \epsilon \|f\|_2 \text{ for all } f \in R(E).$$

The proof is similar to the proof of Theorem 2.1.

One uses the fact that there exists an admissible function  $\bar{\phi}$  such that  $g \cdot \bar{\phi}^{-1} \cdot \bar{\phi}^{-1} \in L^2(E)$ .

### 3. EXAMPLES

EXAMPLE 1. We will construct a compact set  $E$  such that  $0 \in \partial E$ ,  $0$  is a BPE for  $R^2(E)$ , and  $0$  is a limit point of  $\text{Int } E$ . Let  $D = \{|z| \leq 1\}$ . Let  $D_i$ ,  $i = 1, 2, 3, \dots$ , be the open disc centered on the positive real axis at  $3 \cdot 2^{-i-3}$  and having radius  $r_i = \exp(-2^{2i} i^2)$ .

Let  $E = D - \bigcup_{i=1}^{\infty} D_i$ . Then since  $C_2(B(x, r)) \leq F(\log \frac{1}{r})^{-1}$ ,  $r \leq r_0 < 1$ , (see [3]), we have

$$\sum_{n=1}^{\infty} 2^{2n} C_2(A_n - E) = \sum_{n=1}^{\infty} 2^{2n} C(D_n) \leq F \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus  $0$  is a BPE for  $R^2(E)$ . If  $C$  is a positive constant sufficiently small (any positive number  $< \frac{1}{2}$  will do), the set  $\{y \mid y \in \text{Int } E, \text{Dist}(y, \partial E) \geq C |y|\}$  intersects the positive real axis in a sequence of disjoint intervals  $[a_n, b_n]$  such that  $b_n \rightarrow 0$ .

EXAMPLE 2. Next we construct a compact set  $E$  which is like Example 1 in that  $0$  is a limit point of  $\text{Int } E$  and a BPE for  $R^2(E)$ . In this example, however, there exists no sequence  $\{y_n\} \subset \text{Int } E$  such that  $|f(y_n) - f(0)| \leq \epsilon \|f\|_2$  for all  $f \in R(E)$  if  $|y_n| < \delta$ . We will use

important parts of Fernström's construction in [1]. Let  $F$  be a positive constant such that  $C_2(B(z,r)) \leq F(\log \frac{1}{r})^{-1}$  for all  $r, r \leq r_0 < 1$ . Choose  $\alpha, \alpha \geq 1$  such that

$$\frac{F}{\alpha} \cdot \sum_{n=1}^{\infty} \frac{1}{n \log^2 n} < C_2(B(0,1/2)).$$

Let  $A_0$  be the closed unit square with center at 0. Cover  $A_0$  with  $4^n$  squares of side  $2^{-n}$ . Call the squares  $A_n^{(i)}, i = 1, 2, \dots, 4^n$ . In every set  $A_n^{(i)}$  put an open disc  $B_n^{(i)}$  such that  $B_n^{(i)}$  and  $A_n^{(i)}$  have the same center, and the radius of  $B_n^{(i)}$  is  $\exp(-\alpha 4^n n \log^2 n)$ . Let  $D_i, i = 1, 2, 3, \dots$  be an open disc centered on the positive real axis such that  $D_i \subset \{z | 2^{-i-1} \leq |z| \leq 2^{-i}\}$  and  $r_i = \exp(-2^{2i} i^2)$ . For each  $n, n = 1, 2, 3, \dots$ , let  $G_n = \bigcup_i B_n^{(i)}$  where the summation is over those indices  $i$  such that  $1 \leq i \leq 4^n$  and  $B_n^{(i)} \cap (\bigcup_1^{\infty} D_i) = \emptyset$ . Set  $E_1 = A_0 - \bigcup_{n=2}^{\infty} G_n$ . Then  $R^2(E_1)$  has no BPE's in  $\partial E_1$  as is shown in [1].

Now replace a suitable number of the discs

$B_n^{(i)}, B_n^{(i)} \subset \bigcup_{j=2}^{\infty} G_j$ , to obtain a compact set  $E_2$  such that 0 is the only boundary point of  $E_2$  that is a BPE for  $R^2(E_2)$ , (see [1]).

This can be done so that  $\text{Int } E_2 = \bigcup_1^{\infty} D_i$ . If  $y \in \text{Int } E_2$ , let  $\text{norm}(y)$  denote the norm of "evaluation at  $y$ " as a linear functional on  $R^2(E_2)$ . Then if  $\{y_k\} \subset D_i$ , and  $y_k \rightarrow \partial D_i$ ,  $\text{norm}(y_k) \rightarrow \infty$ ; otherwise some point on  $\partial D_i$  would be a BPE for  $R^2(E_2)$ .

For each  $i$  choose an open disc  $D_i' \subset D_i$  such that  $D_i'$  and  $D_i$  are concentric and such that if  $y \in D_i - D_i'$ , then  $\text{norm}(y)$  for the space  $R^2(E_2 - D_i')$  is greater than  $i$ .

Now let  $E = E_2 - \bigcup_{i=1}^{\infty} D_i^1$ .

The radii of the  $D_i^1$  are so small that 0 is also a BPE for  $R^2(E)$ . Let  $\{y_n\}$  be any sequence in  $\text{Int } E$  such that  $y_n \rightarrow 0$ . Let  $\text{norm}(y_n) = \text{norm of "evaluation at } y_n \text{" on } R^2(E)$ . Then for no  $\epsilon > 0$  is there a  $\delta > 0$  such that if  $|y_n| < \delta$ ,  $|f(y_n) - f(0)| \leq \epsilon \|f\|_2$  for all  $f \in R(E)$ .

EXAMPLE 3. Let  $\phi$  be an admissible function. Obtain a compact set  $E$  in the same way that the set  $E_2$  was obtained in Example 2 so that:

- (1)  $D_i$  is centered at  $3 \cdot 2^{-i-2}$  and has radius  $r_i = \phi(3 \cdot 2^{-i-2}) \cdot 2^{-i-2}$
- (2)  $\sum_{n=0}^{\infty} 2^{2n} \cdot \phi(2^{-n})^{-2} C_2(A_n(0) - E) < \infty$ , and
- (3)  $\sum_{n=0}^{\infty} 2^{2n} C_2(A_n(x) - E) = \infty$  for  $x \neq 0$ ,  $x \notin \bigcup D_i$ .

Let  $y_i = 3 \cdot 2^{-i-2}$ . Then by the choice of  $r_i$ ,  $\text{Dist}(y_i, E) \geq 3^{-1} \cdot \phi(|y_i|) |y_i|$ . But there is no  $C > 0$  such that  $\text{Dist}(y_i, \partial E) \geq C |y_i|$  for all  $i$ . Hence Corollary 2.2 applies to the sequence  $\{y_i\}$  but Theorem 2.2 does not.

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