

## INFINITE MATRICES AND ABSOLUTE ALMOST CONVERGENCE

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(Received January 26, 1981 and in revised form August 16, 1982)

**ABSTRACT.** In 1973, Stieglitz [5] introduced a notion of  $F_B$ -convergence which provided a wide generalization of the classical idea of almost convergence due to Lorentz [1]. The concept of strong almost convergence was introduced by Maddox [3] who later on generalized this concept analogous to Stieglitz's extension of almost convergence [4]. In the present paper we define absolute  $F_B$ -convergence which naturally emerges from the concept of  $F_B$ -convergence.

**KEY WORDS AND PHRASES.** Infinite matrices, almost convergence, strong almost convergence,  $F_B$ -convergence, absolute  $F_B$ -convergence.

**1980 MATHEMATICS SUBJECT CLASSIFICATION CODES.** 40C05, 40D05.

### 1. INTRODUCTION.

Let  $\ell_\infty$ ,  $c$ , and  $c_0$  denote respectively the Banach spaces of bounded, convergent, and null sequences  $x = (x_k)$  of complex numbers with norm  $\|x\| = \sup_k |x_k|$ , and let  $v$  be the space of sequences of bounded variation, that is,

$$v = \{x: \|x\| \equiv \sum_{k=0}^{\infty} |x_k - x_{k-1}| < +\infty, x_{-1} = 0\}.$$

Suppose that  $B = (B_i)$  is a sequence of infinite complex matrices with  $B_i = (b_{np}^{(i)})$ . Then  $x \in \ell_\infty$  is said to be  $F_B$ -convergent [5], to the value  $\text{Lim } Bx$ , if

$$\lim_{n \rightarrow \infty} (B_i x)_n = \lim_{n \rightarrow \infty} \sum_{p=0}^{\infty} b_{np}^{(i)} x_p = \text{Lim } Bx,$$

uniformly for  $i = 0, 1, 2, \dots$ .

The space  $F_B$  of  $F_B$ -convergent sequences depends on the fixed chosen sequence  $B = (B_i)$ . In case  $B = B_0 = (I)$  (unit matrix), the space  $F_B$  is same as  $c$  and, for

$B = B_1 = (B_i^{(1)})$ , it is same as the space  $f$  of almost convergent sequences [1], where  $B_i^{(1)} = (b_{np}^{(1)}(i))$  with

$$b_{np}^{(1)}(i) = \left\{ \begin{array}{ll} \frac{1}{n+1}, & i \leq p \leq i+n \\ 0 & \text{elsewhere} \end{array} \right\}$$

Maddox [4] generalized strong almost convergence by saying that  $x_p \rightarrow s[F_B]$  if and only if

$$\sum_p b_{np}^{(1)}(i) |x_p - s| \rightarrow 0 \quad (n \rightarrow \infty, \text{ uniformly in } i) \quad (1.1)$$

assuming that the series in (1.1) converges for each  $n$  and  $i$ .

In particular, if  $B = B_0$ , the  $[F_B] = c$ ; if  $B = B_1$ , then  $[F_B] = [f]$ , the space of strongly almost convergent sequences [3]. We shall write  $e_k = (0, 0, \dots, 0, 1$  ( $k$ th entry),  $0, \dots$ ) and  $e = (1, 1, 1, \dots)$ .

Let  $s$  be the space of all complex sequences and

$$d_B = \{x \in s : \lim_{n \rightarrow \infty} B_n x = \lim_{n \rightarrow \infty} (B_n x)_n \text{ exists for each } i\}$$

$$F_B = \{x \in (d_B \cap l_\infty) : \lim_{n \rightarrow \infty} t_n(i, x) \text{ exists uniformly in } i, \text{ and the limit is independent of } i\},$$

where

$$t_n(i, x) = \left[ \begin{array}{ll} \sum_{p=j}^{\infty} b_{np}^{(1)}(i) x_p, & (n \geq 1) \\ \sum_{p=0}^{\infty} \beta_{0p}(i) x_p, & (n = 0) \\ 0, & (n = -1) \end{array} \right]$$

and

$$\beta_{0p}(i) = \left\{ \begin{array}{ll} 1 & \text{if } p = i, \\ 0 & \text{otherwise.} \end{array} \right\}$$

Let

$$\emptyset_n(i, x) = t_n(i, x) - t_{n-1}(i, x).$$

Therefore, we have

$$\emptyset_n(i, x) = \left[ \begin{array}{ll} \sum_{p=i}^{\infty} [b_{np}^{(1)}(i) - b_{n-1,p}^{(1)}(i)] x_p, & (n \geq 1) \\ \sum_{p=0}^{\infty} \beta_{0p}(i) x_p, & (n = 0) \end{array} \right] \quad (1.2)$$

DEFINITION. Let  $B = (B_i)$  be a sequence of infinite matrices with  $B_i = (b_{np}(i))$ . A sequence  $x \in \ell_\infty$  is said to be absolutely  $F_B$ -convergent if  $\sum_{n=0}^\infty |\theta_n(i,x)|$  converges uniformly for  $i \geq 0$ , and  $\lim_{n \rightarrow \infty} t_n(i,x)$  which must exist should take the same value for all  $i$ . We denote the space of absolute  $F_B$ -convergent sequences by  $v(B)$ .

2. THE MAIN RESULT.

In this note, we denote by  $(v, v(B))$  the set of matrices which give new classes of absolute  $B$ -conservative matrices and absolute almost  $B$ -conservative matrices.

Let  $A$  be any infinite complex matrix for which the  $p$ th row-sum converges for a given  $x$  for all  $x$  in some class.

We have

$$A_p x = (Ax)_p = \sum_{k=0}^\infty a_{pk} x_k$$

and

$$(B_i x)_n = \sum_{p=0}^i b_{np}(i) x_p .$$

Therefore,

$$\begin{aligned} (B_i Ax)_n &= \sum_{p=0}^\infty b_{np}(i) A_p x \\ &= \sum_{p=0}^\infty b_{np}(i) \sum_{k=0}^\infty a_{pk} x_k, \end{aligned}$$

and, assuming the interchange of order of summation can be justified (see lemma), we get that

$$(B_i Ax)_n = \sum_{k=0}^\infty \sum_{p=0}^i b_{np}(i) a_{pk} x_k \tag{2.1}$$

Now, by (1.2) and (2.1), we have

$$\begin{aligned} \theta_n(i, Ax) &= t_n(i, Ax) - t_{n-1}(i, Ax) \\ &= \begin{cases} \sum_{p=0}^i [b_{np}(i) - b_{n-1,p}(i)] A_p x, & (n \geq 1), \\ \sum_{p=0}^i \beta_{0p}(i) A_p x & , \quad (n = 0), \end{cases} \\ &= \sum_{k=0}^\infty g_{nk}(i) x_k, \end{aligned} \tag{2.2}$$

where

$$g_{nk}^{(i)} = \begin{cases} \sum_{p=0}^{\infty} [b_{np}^{(i)} - b_{n-1,p}^{(i)}] a_{pk}, & (n \geq 1), \\ \sum_{p=0}^{\infty} \beta_{0p}^{(i)} a_{pk}, & (n = 0). \end{cases}$$

**THEOREM.** Let  $B = (B_i)$  be a sequence of infinite matrices with

$$\sup_n \sum_{p=0}^{\infty} |b_{np}^{(i)}| < \infty, \quad \text{for each } i.$$

Let  $A$  be an infinite matrix. Then  $A: v \rightarrow v(B)$  if and only if

- (i)  $\sup_{p,k} \left| \sum_{\ell=k}^{\infty} a_{p\ell} \right| < \infty,$
- (ii) there is an  $N$  such that for  $r, i = 0, 1, 2, \dots$ 

$$\sum_{n=N}^{\infty} \left| \sum_{k=0}^r g_{nk}^{(i)} \right| \leq K \quad (\text{constant}),$$
- (iii)  $(a_{pk})_{p \geq 0} \in v(B)$  for each  $k$ , and
- (iv)  $(\sum_{k=0}^{\infty} a_{pk})_{p \geq 0} \in v(B).$

Let  $A \in (v, v(B))$ . For each  $k$ , let  $a_{pk}$  be  $F_B$ -convergent with limit  $\alpha_k$ . And let  $\sum_{k=0}^{\infty} a_{pk}$  be  $F_B$ -convergent with limit  $\alpha$ . (In each case, limit is taken for  $p \geq 0$ ).

If  $x = (x_k) \in v$ , then

$$\lim_{n \rightarrow \infty} t_n(i, Ax) = \alpha \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} (x_k - \lim_{k \rightarrow \infty} x_k) \alpha_k.$$

We use the following lemma in the proof.

**LEMMA.** If either the necessity part or the sufficiency part of the theorem holds, then, for  $x \in v$ ,

$$\sum_{p=0}^{\infty} b_{np}^{(i)} \sum_{k=0}^{\infty} a_{pk} x_k = \sum_{k=0}^{\infty} x_k \sum_{p=0}^{\infty} b_{np}^{(i)} a_{pk}.$$

**PROOF.** If either  $A: v \rightarrow v(B)$  or the conditions (i)-(iv) of the theorem hold, then by partial summation, for  $x \in v$ ,

$$\sum_{k=0}^{\infty} a_{pk} x_k = \sum_{k=0}^{\infty} d_{pk} (x_k - x_{k-1})$$

where  $d_{pk} = \sum_{\ell=k}^{\infty} a_{p\ell}$ . Since condition (i) holds,  $d_{pk}$  is bounded for all  $p, k$ . Thus

$$\begin{aligned} \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} a_{pk} x_k &= \sum_{p=0}^{\infty} b_{np}(i) \sum_{k=0}^{\infty} d_{pk} (x_k - x_{k-1}) \\ &= \sum_{k=0}^{\infty} (x_k - x_{k-1}) \sum_{p=0}^{\infty} b_{np}(i) d_{pk}, \end{aligned}$$

(where the inversion is justified by absolute convergence)

$$= \sum_{k=0}^{\infty} x_k \sum_{p=0}^{\infty} b_{np}(i) a_{pk}$$

since

$$\lim_{k \rightarrow \infty} x_k \sum_{p=0}^{\infty} b_{np}(i) d_{pk} = 0.$$

PROOF OF THEOREM. Necessity. Condition (i) follows from the fact that  $A: v \rightarrow \ell_{\infty}$ .

Since  $e_k, e \in v$ , necessity of (iii) and (iv) is obvious.

It is clear that, for fixed  $p$  and  $j$ ,

$$x \rightarrow \sum_{k=0}^j a_{pk} x_k$$

is a continuous linear functional on  $v$ . We are given that, for all  $x \in v$ , it tends to a limit as  $j \rightarrow \infty$  (for fixed  $p$ ) and hence, by the Banach-Steinhaus Theorem [2], this limit  $A_p x$  is also a continuous linear functional on  $v$ .

We observe that, although  $\sum_{n=0}^{\infty} |\phi_n(i, Ax)|$  is uniformly convergent in  $i$ , it needs not be uniformly bounded in  $i$ . For example, if  $\phi_0(i, Ax) = i$  and  $\phi_n(i, Ax) = 0$  ( $n \geq 1$  and  $i$ ), then  $\sum_{n=0}^{\infty} |\phi_n(i, Ax)|$  is uniformly convergent in  $i \geq 0$  but not uniformly bounded. Now, we can say that uniform convergence bears only on the behaviour of  $\phi_n(i, Ax)$  for sufficiently large  $n$ . Thus, by definition, there is an  $m$  such that

$$q_{m,i}(x) = \sum_{n=m}^{\infty} |\phi_n(i, Ax)|.$$

For  $m \geq 0, i \geq 0, q_{m,i}$  is a continuous seminorm on  $v$ , and there is an integer  $N$  such that  $\{q_{N,i}\}_{i \geq 0}$  is pointwise bounded on  $v$ . Such an  $N$  exists. For suppose not. Then for  $r = 0, 1, 2, \dots$  there exists  $x_r \in v$  with

$$\sup_{i \geq 0} q_{r,i}(x_r) = \infty.$$

By the principle of condensation of singularities [6],

$$\{x \in v: \sup_{i \geq 0} q_{r,i}(x) = \infty \text{ for } r = 0, 1, 2, \dots\}$$

is of second category in  $v$  and hence nonempty, i.e., there is  $x \in v$  with

$$\sup_{i \geq 0} q_{r,i}(x) = \infty \quad \text{for } r = 0, 1, 2, \dots$$

But this contradicts the fact that to each  $x \in v$  there exists an integer  $N_x$  with

$$\sup_{i \geq 0} q_{N_x, i}(x) < \infty.$$

Now, by another application of the Banach-Steinhaus Theorem, there exists a constant  $M$  such that

$$q_{N,i}(x) \leq M \|x\|. \quad (2.3)$$

Apply (2.3) with  $x = (x_k)$  defined by  $x_k = 1$  for  $k \leq r$  and 0 for  $k > r$ . Hence (ii) must hold.

Sufficiency. Suppose that the conditions (i)-(iv) hold and that  $x \in v$ . We have defined  $v(B)$  as a subspace of  $\ell_\infty$ . Thus, in order to show that  $Ax \in v(B)$ , it is necessary to prove that  $Ax$  is bounded. By virtue of condition (i), this follows immediately.

Now, it follows from (iv) and the lemma that

$$\sum_{k=0}^{\infty} g_{nk}(i)$$

converges for all  $i, n$ . Hence, if we write

$$h_{nk}(i) = \sum_{\ell=k}^{\infty} g_{n\ell}(i),$$

then  $h_{nk}(i)$  is defined, also for fixed  $i, n$ ,

$$h_{nk}(i) \rightarrow 0 \quad (2.4)$$

as  $k \rightarrow \infty$ . Now condition (iv) gives us that

$$\sum_{n=0}^{\infty} |h_{n0}(i)| \quad (2.5)$$

converges uniformly in  $i$ , and, for suitable chosen  $N$ ,

$$\sum_{n=N}^{\infty} |h_{n0}(i)| \quad (2.6)$$

is bounded. By virtue of condition (iii), for fixed  $k$ , we get that

$$\sum_{n=0}^{\infty} |g_{nk}(i)|$$

converges uniformly in  $i$ . Since

$$h_{nk}(i) = h_{n0}(i) - \sum_{\ell=0}^{k-1} g_{n\ell}(i), \quad (2.7)$$

it follows that, for fixed  $k$ ,

$$\sum_{n=0}^{\infty} |h_{nk}(i)| \tag{2.8}$$

converges uniformly in  $i$ .

Now

$$\begin{aligned} \phi(i, Ax) &= \sum_{k=0}^{\infty} g_{nk}(i) x_k \\ &= \sum_{k=0}^{\infty} [h_{nk}(i) - h_{n,k+1}(i)] x_k \\ &= \sum_{k=0}^{\infty} h_{nk}(i) (x_k - x_{k-1}), \end{aligned} \tag{2.9}$$

by (2.4) and the boundedness of  $x_k$ .

Condition (ii) and the boundedness of (2.6) show that

$$\sum_{n=N}^{\infty} |h_{nk}(i)| \tag{2.10}$$

is bounded for all  $k, i$ . We can make

$$\sum_{k=k_0+1}^{\infty} |x_k - x_{k-1}|$$

arbitrarily small by choosing  $k_0$  sufficiently large. It therefore follows that, given  $\epsilon > 0$ , we can choose  $k_0$  so that, for all  $i$ ,

$$\sum_{n=N}^{\infty} \left| \sum_{k=k_0+1}^{\infty} h_{nk}(i)(x_k - x_{k-1}) \right| < \epsilon. \tag{2.11}$$

By the uniform convergence of (2.8), it follows that, once  $k_0$  has been chosen, we can choose  $n_0$  so that, for all  $i$ ,

$$\sum_{n=n_0+1}^{\infty} \left| \sum_{k=0}^{k_0} h_{nk}(i)(x_k - x_{k-1}) \right| < \epsilon.$$

It follows from (2.11) that the same inequality holds when  $\sum_{n=N}^{\infty}$  is replaced by  $\sum_{n=n_0+1}^{\infty}$ ; hence, for all  $i$ ,

$$\sum_{n=n_0+1}^{\infty} \left| \sum_{k=0}^{\infty} h_{nk}(i)(x_k - x_{k-1}) \right| < 2\epsilon. \tag{2.12}$$

Hence,

$$\sum_{n=n_0+1}^{\infty} |\phi_n(i, Ax)| < 2\epsilon.$$

Thus

$$\sum_{n=0}^{\infty} |\phi_n(i, Ax)|$$

converges uniformly.

Now, by virtue of (2.9), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} t_n(i, Ax) - t_{N-1}(i, Ax) &= \sum_{n=N}^{\infty} \sum_{k=0}^{\infty} h_{nk}(i) (x_k - x_{k-1}) \\ &= \sum_{k=0}^{\infty} (x_k - x_{k-1}) \sum_{n=N}^{\infty} h_{nk}(i) \end{aligned} \quad (2.13)$$

the assertion being justified by absolute convergence because of the boundedness of (2.10). By (2.7), we have

$$\begin{aligned} \sum_{n=N}^{\infty} h_{nk}(i) &= \sum_{n=N}^{\infty} h_{no}(i) - \sum_{\ell=0}^{k-1} \sum_{n=N}^{\infty} g_{n\ell}(i) \\ &= \alpha - \sum_{\ell=0}^{k-1} \alpha_{\ell} - \sum_{\ell=k}^{\infty} \sum_{p=0}^{\infty} b_{N-1,p}(i) a_{p\ell}. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} t_n(i, Ax) = \alpha \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} (x_k - \lim_{k \rightarrow \infty} x_k) \alpha_k.$$

This completes the proof.

#### REFERENCES

1. LORENTZ, G.G. A contribution to the theory of divergent series, Acta Math. 80 (1948), 167-190.
2. MADDOX, I.J. Elements of Functional Analysis, Cambridge University Press, 1970.
3. MADDOX, I.J. A new type of convergence, Math. Proc. Camb. Phil. Soc. 83 (1978), 61-64.
4. MADDOX, I.J. On strong almost convergence, Math. Proc. Camb. Phil. Soc. 85 (1979), 345-350.
5. STIEGLITZ, M. Eine Verallgemeinerung des Begriffs der Fastkonvergenz, Math. Japon. 18 (1973), 53-70.
6. YOSIDA, K. Functional Analysis, Springer-Verlag, Berlin-Heidelberg-New York, 1966.

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