

## A NOTE ON A PAPER BY S. HABER

A. McD. MERCER

Department of Mathematics and Statistics  
University of Guelph  
Guelph, Ontario, Canada N1G 2W1

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**ABSTRACT.** A technique used by S. Haber to prove an elementary inequality is applied here to obtain a more general inequality for convex sequences.

**KEY WORDS AND PHRASES.** Convex sequences, Hadamard's inequality for convex functions, rearrangements.

**1980 MATHEMATICS SUBJECT CLASSIFICATION CODE.** Primary 26D15, Secondary 40G05.

### 1. INTRODUCTION.

Let  $a$  and  $b$  be non-negative. Then the following elementary inequality was proved in [1].

$$\frac{1}{n+1}[a^n + a^{n-1}b + \dots + b^n] \geq \left(\frac{a+b}{2}\right)^n \quad (n=0,1,2,\dots) \dots \quad (1.1)$$

Now this inequality can be obtained at once by taking  $f(t) = t^n$  in the well-known result

$$\frac{1}{b-a} \int_a^b f(t) dt \geq f\left(\frac{a+b}{2}\right) \dots \dots \quad (1.2)$$

which holds whenever  $f$  is convex in  $[a,b]$ . However, the method used in [1] to obtain (1.1) is interesting and it is the purpose of the present note to show that it can be used to prove a considerably more general result about sequences. Indeed this more general result will have (1.2) as a consequence.

### 2. MAIN RESULTS.

A lemma which we shall use is the following

**LEMMA.** If

$$\beta_0 \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_m$$

and

$$\sum_{\nu=0}^m \alpha_\nu = 0$$

and if the ordering of the  $\alpha_\nu$  is such that each positive  $\alpha$  precedes all the negative ones, then

$$\sum_{\nu=0}^m \alpha_\nu \beta_\nu \geq 0 .$$

This lemma, which is easily proved, is not the one stated by Haber but, essentially, it is what he used. For with  $b_i$  defined as in [1]

$$(i = 0 \ 1 \ 2, \dots, [\frac{n}{2}]: n \text{ even})$$

we do *not* in fact have

$$\sum_{i=0}^{[\frac{n}{2}]} b_i = 0$$

which is what is needed to apply the lemma quoted there.

Our result is the following.

THEOREM. Let  $\{u_\nu\}_{\nu=0}^n$  be a convex sequence. Then

$$\frac{1}{n+1} \sum_{\nu=0}^n u_\nu \geq \frac{1}{2^n} \sum_{\nu=0}^n \binom{n}{\nu} u_\nu \dots \tag{2.1}$$

To see that (1.2) is a consequence of (2.1) let the function  $f(x)$  be bounded and convex (and hence continuous) on  $[a,b]$  and take

$$u_\nu = f(a + \frac{\nu}{n} (b-a)) .$$

Then (2.1) reads

$$\frac{1}{n+1} \sum_{\nu=0}^n f(a + \frac{\nu}{n} (b-a)) \geq \frac{1}{2^n} \sum_{\nu=0}^n \binom{n}{\nu} f(a + \frac{\nu}{n} (b-a)) \dots \tag{2.2}$$

On letting  $n \rightarrow \infty$  the left-hand side here tends to the left-hand side of (1.2). And by virtue of Bernstein's result

$$\lim_{n \rightarrow \infty} \sum_{\nu=0}^n \binom{n}{\nu} \phi(\frac{\nu}{n}) x^\nu (1-x)^{n-\nu} = \phi(x) \dots \tag{2.3}$$

whenever  $\phi \in C[0,1]$  we see that the right-hand side of (2.2) tends to  $f(\frac{a+b}{2})$ . Merely take  $\phi(x) = f(a + x(b-a))$  and  $x = 1/2$  in (2.3).

We now proceed to prove (2.1).

PROOF. Following Haber let us put  $Q = [\frac{n}{2}]$  and write

$$\sum_{\nu=0}^Q * \gamma_\nu = \begin{cases} \gamma_0 + \gamma_1 + \dots + \gamma_Q & \text{if } n \text{ is odd} \\ \gamma_0 + \gamma_1 + \dots + \gamma_{Q-1} + \frac{1}{2} \gamma_Q & \text{if } n \text{ is even} . \end{cases}$$

Then

$$\frac{1}{n+1} \sum_{\nu=0}^n u_\nu - \frac{1}{2^n} \sum_{\nu=0}^n \binom{n}{\nu} u_\nu = \sum_{\nu=0}^Q * c_\nu [u_\nu + u_{n-\nu}]$$

where

$$c_v = \frac{1}{n+1} - \frac{1}{2^n} \binom{n}{v}$$

Since  $\{u_v\}_{v=0}^n$  is convex then

$$u_{v+1} + u_{n-v-1} \leq u_v + u_{n-v} \quad (0 \leq v \leq n-1)$$

which is to say that the sequence  $\{u_v + u_{n-v}\}_{v=0}^n$  is non-increasing. We see too that the sequence  $\{c_v\}_{v=0}^n$  is non-increasing and that  $\sum_{v=0}^n c_v = 0$ . Appealing to the Lemma quoted above we find that

$$\sum_{v=0}^Q c_v [u_v + u_{n-v}] \geq 0$$

and this completes the proof of (2.1).

In conclusion I wish to thank the referee for his helpful advice concerning the lemma used here.

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