Internat. J. Math. & Math. Sci.
Vol. 6 No. 3 (1983) 449-458

SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

E.M. SILVIA

Department of Mathematics
University of California, Davis
Davis, California 95616

(Received January 24, 1983)

ABSTRACT. Let $\mathcal{K}[C,D]$, $-1 \leq D < C \leq 1$, denote the class of functions $g(z)$, $g(0) = g'(0) - 1 = 0$, analytic in the unit disk $U = \{z: |z| < 1\}$ such that $1 + (zg''(z)/g'(z))$ is subordinate to $(1+Cz)/(1+Dz)$, $z \in U$. We investigate the subclasses of close-to-convex functions $f(z)$, $f(0) = f'(0) - 1 = 0$, for which there exists $g \in \mathcal{K}[C,D]$ such that f'/g' is subordinate to $(1+Az)/(1+Bz)$, $-1 \leq B < A \leq 1$. Distortion and rotation theorems and coefficient bounds are obtained. It is also shown that these classes are preserved under certain integral operators.

KEY WORDS AND PHRASES. Univalent, convex, starlike, subordination, convolution.

AMS (MOS) SUBJECT CLASSIFICATION (1980) CODES: 30C45, 30C55.

1. INTRODUCTION.

Let S denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic and univalent in the unit disk $U = \{z: |z| < 1\}$. For functions g and G analytic in U we say that g is subordinate to G, denoted $g \prec G$, if there exists a Schwarz function $w(z)$, $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U, such that $g(z) = G(w(z))$. If G is univalent in U then $g \prec G$ if and only if $g(0) = G(0)$ and $g(U) \subset$

$G(U)$. For A and B, $-1 \leq B < A \leq 1$, a function p analytic in U with $p(0) = 1$ is in the class $\mathcal{P}[A,B]$ if $p(z) \prec (1+Az)/(1+Bz)$. This class was introduced by Janowski [4]. Given C and D, $-1 \leq D < C \leq 1$, $\mathcal{K}[C,D]$ and $\mathcal{S}^*[C,D]$ denote the classes of functions f analytic in U with $f(0) = f'(0) - 1 = 0$ such that $1 + zf''(z)/f'(z) \in \mathcal{P}[C,D]$ and $zf'(z)/f(z) \in \mathcal{P}[C,D]$, respectively. The classes $\mathcal{S}^*[C,D]$ were introduced by Janowski [4] and studied further by Goel and Mehrok ([1] and [3]). For $C = 1$ and $D = -1$, $\mathcal{K}[1,-1] = \mathcal{K}$ ($\mathcal{S}^*[1,-1] = \mathcal{S}^*$), the well-known subclass of convex (starlike) functions.

A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in U is said to be in the class $C[A,B;C,D]$, $-1 \leq B < A \leq 1$, $-1 \leq D < C \leq 1$, if there exists $g \in \mathcal{K}[C,D]$ such that $f'/g' \in \mathcal{P}[A,B]$. The well-known (Kaplan [5]) class of close-to-convex functions is $C[1,-1;1,-1] = \mathcal{C}$ while $\mathcal{K}[C,D] \subset \mathcal{K}$ and $\mathcal{P}[A,B] \subset \mathcal{P}[1,-1]$ shows $C[A,B;C,D] \subset \mathcal{C} \subset S$. Since $g \in \mathcal{S}^*[C,D]$ if and only if $\int_0^z g(\zeta)\zeta^{-1}d\zeta \in \mathcal{K}[C,D]$, we also note that $C[1,-1;C,D]$ was studied by Goel and Mehrok ([2] and [3]).

In Section 2 of this paper we obtain distortion and rotation theorems for $f'(z)$ whenever $f \in C[A,B;C,D]$ and a subordination result relating $C[A,B;C,D]$ and $\mathcal{P}[A,B]$. In Section 3, it is shown that the class $C[A,B;C,D]$ is preserved under certain integral operators. We conclude with coefficient inequalities.

2. DISTORTION AND ROTATION THEOREMS.

Unless otherwise mentioned in the sequel, the only restrictions on the real constants A, B, C and D are that $-1 \leq D < C \leq 1$ and $-1 \leq B < A \leq 1$.

THEOREM 1. *For* $f \in C[A,B;C,D]$, $|z| \leq r < 1$,

$$\frac{(1-Ar)(1-Dr)^{(C-D)/D}}{1-Br} \leq |f'(z)| \leq \frac{(1+Ar)(1+Dr)^{(C-D)/D}}{1+Br} \quad , \quad D \neq 0$$

and

$$\frac{(1-Ar)\exp(-Cr)}{1-Br} \leq |f'(z)| \leq \frac{(1+Ar)\exp(Cr)}{1+Br} \quad , \quad D = 0.$$

The bounds are sharp.

PROOF. For $f \in C[A,B;C,D]$, there exists a $g \in K[C,D]$ and $p \in P[A,B]$ such that

$$f'(z) = g'(z)p(z). \tag{2.1}$$

Since $g \in K[C,D]$ if and only if $zg' \in S^*[C,D]$, for $|z| \leq r < 1$ [4]

$$\begin{aligned} &(1-Dr)^{(C-D)/D} \leq |g'(z)| \leq (1+Dr)^{(C-D)/D} \quad , \quad D \neq 0, \text{ and} \\ &\exp(-Cr) \leq |g'(z)| \leq \exp(Cr) \quad , \quad D = 0. \end{aligned} \tag{2.2}$$

For $p \in P[A,B]$, $|z| \leq r$, the univalence of $(1+Az)/(1+Bz)$ gives

$$\frac{1-Ar}{1-Br} \leq |p(z)| \leq \frac{1+Ar}{1+Br}. \tag{2.3}$$

The result follows immediately upon applying (2.3) and (2.2) to (2.1).

Equality is obtained for $f \in C[A,B;C,D]$ satisfying

$$f'(z) = \begin{cases} \dfrac{(1+Az)(1+Dz)^{(C-D)/D}}{1+Bz} & , \quad D \neq 0 \\ \\ \dfrac{(1+Az)\exp(Cz)}{1+Bz} & , \quad D = 0 \end{cases} \tag{2.4}$$

and $z = \pm r$.

REMARK. For $A = 1$ and $B = -1$, Theorem 1 agrees with Theorem 3 of Goel and Mehrok [2].

THEOREM 2. *For* $f \in C[A,B;C,D]$, $|z| \leq r < 1$,

$$|\arg f'(z)| \leq \begin{cases} \frac{C-D}{D} \arcsin(Dr) + \arcsin \frac{(A-B)r}{1-ABr^2} & , D \neq 0 \\ \arcsin(Cr) + \arcsin \frac{(A-B)r}{1-ABr^2} & , D = 0. \end{cases}$$

PROOF. From (2.1) we have

$$|\arg f'(z)| \leq |\arg g'(z)| + |\arg p(z)|. \tag{2.5}$$

Since $zg' \in \mathcal{S}^*[C,D]$, we know [2] that for $|z| \leq r < 1$

$$|\arg g'(z)| \leq \begin{cases} \frac{C-D}{D} \arcsin(Dr) & , D \neq 0 \\ Cr & , D = 0. \end{cases} \tag{2.6}$$

For $p \in \mathcal{P}[A,B]$, $p(|z| < r)$ is contained in the disk $\left|w - \frac{1-ABr^2}{1-Br^2}\right| < \frac{(A-B)r}{1-B^2r^2}$ from which it follows that

$$|\arg p(z)| \leq \arcsin \frac{(A-B)r}{1-ABr^2}. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5) gives the result.

REMARKS 1. For $A = 1$, $B = -1$, Theorem 2 agrees with Theorem 4 of Goel and Mehrok [2].

2. For $A = C = 1$, $B = D = -1$, Theorem 2 reduces to the result of Krzyz [7] that

$$|\arg f'(z)| \leq 2(\arcsin r + \arctan r), \quad |z| \leq r < 1.$$

The *convolution* or *Hadamard product* of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. In order to obtain a subordination result linking $C[A,B;C,D]$ and $\mathcal{P}[A,B]$ we need the following

LEMMA A (Ruscheweyh and Sheil-Small, [11]). *Let* φ *and* ψ *be convex in* U *and suppose* $f \prec \psi$. *Then* $\varphi * f \prec \varphi * \psi$.

THEOREM 3. *If* $f \in C[A,B;C,D]$ *then there exists* $p \in \mathcal{P}[A,B]$ *such that for all* s *and* t *with* $|s| \le 1$, $|t| \le 1$,

$$\frac{f'(sz)p(tz)}{f'(tz)p(sz)} \prec \begin{cases} \left(\frac{1+Dsz}{1+Dtz}\right)^{(C-D)/D} & , D \neq 0 \\ \exp[C(s-t)z] & , D = 0. \end{cases} \tag{2.8}$$

PROOF. We will use an approach due to Ruscheweyh [10]. From (2.1) we have $\frac{zf''(z)}{f'(z)} = \frac{zg''(z)}{g'(z)} + \frac{zp'(z)}{p(z)}$ for $g \in \mathcal{K}[C,D]$ and $p \in \mathcal{P}[A,B]$. Therefore,

$$\frac{zf''(z)}{f'(z)} - \frac{zp'(z)}{p(z)} = \left(1 + \frac{zg''(z)}{g'(z)}\right) - 1 \prec \frac{(C-D)z}{1 + Dz}. \tag{2.9}$$

For s and t such that $|s| \le 1$, $|t| \le 1$, the function

$h(z) = \int_0^z \left(\frac{s}{1-su} - \frac{t}{1-tu}\right)du$ is convex in U. Applying Lemma A to (2.9)

with this h, we have

$$\left(\frac{zf''(z)}{f'(z)} - \frac{zp'(z)}{p(z)}\right) * h(z) \prec \frac{(C-D)z}{1 + Dz} * h(z). \tag{2.10}$$

Given any function $\ell(z)$ analytic in U with $\ell(0) = 0$, we have

$(\ell * h)(z) = \int_{tz}^{sz} \ell(u)\,\frac{du}{u}$, $z \in U$, so that (2.10) reduces to

$$\log\left(\frac{f'(sz)p(tz)}{p(sz)f'(tz)}\right) \prec (C-D)\int_{tz}^{sz}\frac{du}{1 + Du}\,. \tag{2.11}$$

Integrating the righthand side of (2.11) and exponentiating both sides leads to (2.8).

COROLLARY 1. _If_ $f \in C[A,B;C,D]$ _then there exists a_ $p \in \mathcal{P}[A,B]$ _and a Schwarz function_ $w(z)$ _such that_

$$f'(z) = \begin{cases} p(z)(1 + Dw(z))^{(C-D)/D} & , D \neq 0 \\ p(z)\exp(Cw(z)) & , D = 0. \end{cases}$$

PROOF. The result follows directly upon substituting $s = 1$ and $t = 0$ into Theorem 3.

COROLLARY 2. _If_ $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C[A,B;C,D]$ _then_ $|a_2| \leq \frac{(C-D)+(A-B)}{2}$.

PROOF. If $g \prec F$ then $|g'(0)| \leq |F'(0)|$ [8]. From Corollary 1, we take $g(z) = f'(z)/p(z)$ and

$$F(z) = \begin{cases} (1+Dz)^{(C-D)/D} & , D \neq 0 \\ \exp(Cz) & , D = 0 \end{cases}.$$

Then $g'(0) = 2a_2 - c_1$ for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $F'(0) = C - D$. Therefore $2|a_2| - |c_1| \leq |C-D|$ and $|a_2| \leq \frac{(C-D)+|c_1|}{2} \leq \frac{(C-D)+(A-B)}{2}$ as claimed.

3. INVARIANCE PROPERTIES.

We will need the following lemmas.

LEMMA B (Ruscheweyh and Sheil-Small, [11]). *Let* φ *be convex and* g *starlike in* U. *Then for* F *analytic in* U *with* $F(0) = 1$, $\frac{\varphi * Fg}{\varphi * g}(U)$ *is contained in the convex hull of* $F(U)$.

LEMMA C (Silverman and Silvia, [12]). *If* $g \in \mathcal{S}^*[C,D]$ *then for* $\varphi \in \mathcal{K}$, $\varphi * g \in \mathcal{S}^*[C,D]$.

THEOREM 4. *If* $\varphi \in \mathcal{K}$, *and* $f \in C[A,B;C,D]$ *then* $\varphi * f \in C[A,B;C,D]$.

PROOF. For $f \in C[A,B;C,D]$ there exists $g \in \mathcal{S}^*[C,D]$ and $F \in \mathcal{P}[A,B]$ such that $zf'(z) = g(z)F(z)$. Since $(1+Az)/(1+Bz)$ is convex in U, by Lemma B,

$$\frac{z(\varphi * f)'}{\varphi * g} = \frac{\varphi * Fg}{\varphi * g} \prec \frac{1+Az}{1+Bz} \qquad (z \in U) \tag{3.1}$$

for $\varphi \in \mathcal{K}$. From Lemma C, $\varphi * g \in \mathcal{S}^*[C,D]$ so that (3.1) is equivalent to $\varphi * f \in C[A,B;C,D]$.

REMARK. For $A = C = 1$, $B = D = -1$, Theorem 4 was proved by Ruscheweyh and Sheil-Small [11].

COROLLARY. *If* $f \in C[A,B;C,D]$ *then so are*

(i) $F_1(z) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t)\,dt \quad , \quad \text{Re } \gamma > 0$

and

(ii) $F_2(z) = \int_0^z \frac{f(\zeta) - f(x\zeta)}{\zeta - x\zeta}\, d\zeta, \qquad |x| \le 1,\ x \ne 1.$

PROOF. Observe that $F_j(z) = (h_j * f)(z)$, $j = 1,2$, where $h_1(z) = \sum_{n=1}^{\infty} \frac{1+\gamma}{n+\gamma} z^n$, $\operatorname{Re} \gamma > 0$, and $h_2(z) = \sum_{n=1}^{\infty} \frac{1-x^n}{(1-x)n} z^n = \frac{1}{1-x} \log[\frac{1-xz}{1-z}]$, $|x| \leq 1$, $x \neq 1$. Since h_1 was shown to be convex, by Ruscheweyh [9] and h_2 is clearly convex, the result follows immediately from Theorem 4.

REMARK. Goel and Mehrok [3] showed that $C[1,-1;C,D]$ was preserved under $F_1(z)$ when $\gamma = 1,2,3,...$ and under $F_2(z)$ when $x = -1$ by a different method.

4. COEFFICIENT INEQUALITIES.

We begin with coefficient inequalities for $\mathcal{K}[C,D]$.

LEMMA. *For* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{K}[C,D]$ *and* μ *complex*

$$|b_2| \leq \frac{C-D}{2}, \quad \textit{and}$$

$$|b_3 - \mu b_2^2| \leq \frac{C-D}{6} \max\{1, |\tfrac{3}{2} \mu(C-D)-(C-2D)|\}.$$

PROOF. For $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{K}[C,D]$, there exists a Schwarz function $w(z) = \sum_{n=1}^{\infty} \gamma_n z^n$ such that $1 + (zg''(z)/g'(z)) = (1+Cw(z))/(1+Dw(z))$ or $zg''(z)/g'(z) = (C-D)w(z)/(1+Dw(z))$. Substitution of the series expansions and comparison of the coefficients leads to

$$b_2 = \frac{C-D}{2} \gamma_1 \quad \text{and} \quad b_3 = \frac{C-D}{6} \{\gamma_2 + (C-2D)\gamma_1^2\}.$$

Therefore, $|b_2| \leq \frac{C-D}{2}$ and

$$b_3 - \mu b_2^2 = \frac{C-D}{6} \{\gamma_2 + [(C-2D) - \tfrac{3}{2} \mu(C-D)]\gamma_1^2\}. \tag{4.1}$$

We know [6] that for s complex

$$|\gamma_2 - s\gamma_1^2| \le \max\{1, |s|\}. \tag{4.2}$$

Combining (4.2) and (4.1) yields the result.

REMARK. If we apply the inequality $|\gamma_2| \le 1 - |\gamma_1|^2$ [8] to (4.1), the same proof shows that

$$|b_3 - \mu b_2^2| \le \frac{C-D}{6} + \frac{2}{3(C-D)}\{|(C-2D) - \frac{3}{2}\mu(C-D)| - 1\}|b_2|^2.$$

THEOREM 5. *For* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C[A,B;C,D]$

$$|a_2| \le \frac{(C-D)+(A-B)}{2} \quad \text{and}$$

$$|a_3| \le \begin{cases} \dfrac{C-D}{6} + \dfrac{(A-B)(C-D+1)}{3}, & |C-2D| \le 1 \\[2ex] \dfrac{(C-D)(C-2D)}{6} + \dfrac{(A-B)(C-D+1)}{3}, & |C-2D| > 1. \end{cases}$$

PROOF. There exists a $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{K}[C,D]$ and a Schwarz function $w(z) = \sum_{n=1}^{\infty} \gamma_n z^n$ such that $f'(z)/g'(z) = (1+Aw(z))/(1+Bw(z))$, $z \in U$. Comparing series expansions, we see $a_2 = b_2 + \frac{A-B}{2}\gamma_1$ and

$$a_3 = b_3 + \frac{2}{3}(A-B)b_2\gamma_1 + \frac{(A-B)}{3}(\gamma_2 - B\gamma_1^2). \tag{4.3}$$

The bound for $|a_2|$ follows from the Lemma. Applying (4.2) and the Lemma ($\mu = 0$) to (4.3), we have

$$|a_3| \le \frac{C-D}{6}\max\{1,|C-2D|\} + \frac{(A-B)(C-D)}{3} + \frac{(A-B)}{3}\max\{1,|B|\}$$

$$= \frac{C-D}{6}\max\{1,|C-2D|\} + \frac{(A-B)(C-D+1)}{3}.$$

REFERENCES

1. Goel, R. M. and Mehrok, B. S., On the coefficients of a subclass of starlike functions. Indian J. Pure Appl. Math., 12(1981), 634-647.

2. Goel, R. M. and Mehrok, B. S., On a class of close-to-convex functions. Indian J. Pure Appl. Math., 12(1981), 648-658.

3. Goel, R. M. and Mehrok, B. S., Some invariance properties of a subclass of close-to-convex functions. Indian J. Pure Appl. Math., 12(1981), 1240-1249.

4. Janowski, W., Some extremal problems for certain families of analytic functions. Ann. Polon. Math., 28(1973), 297-326.

5. Kaplan, W., Close-to-convex schlicht functions. Mich. Math. J., 1(1952), 169-185.

6. Keogh, F. R. and Merkes, E. P., A coefficient inequality for certain classes of analytic functions. Proc. Amer. Math. Soc., 20(1969), 8-12.

7. Krzyz, J., On the derivative of close-to-convex functions. Colloq. Math., 10(1963), 143-146.

8. Nehari, Z., Conformal Mapping. McGraw-Hill, New York, 1952.

9. Ruscheweyh, St., New criteria for univalent functions. Proc. Amer. Math. Soc., 49(1975), 109-115.

10. Ruscheweyh, St., A subordination theorem for Φ-like functions. J. London Math. Soc., 13(1976), 275-280.

11. Ruscheweyh, St. and Sheil-Small, T., Hadamard products of schlicht functions and the Polyà-Schoenberg conjecture. Comm. Math. Helv. 48(1973), 119-135.

12. Silverman, H. and Silvia, E. M., Subclasses of starlike functions subordinate to convex functions (submitted).