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SUPREMUM NORM DIFFERENTIABILITY

I.E. LEONARD and K.F. TAYLOR

Department of Mathematics, University of Saskatchewan,
Saskatoon, Saskatchewan, Canada S7N 0W0

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ABSTRACT. The points of Gateaux and Fréchet differentiability of the norm in $C(T, E)$ are obtained, where T is a locally compact Hausdorff space and E is a real Banach space. Applications of these results are given to the space $\ell_\infty(E)$ of all bounded sequences in E, and to the space $B(\ell_1, E)$ of all bounded linear operators from ℓ_1 into E.

KEY WORDS AND PHRASES. Banach spaces, continuous functions, vector-valued functions, supremum norm, smooth points.

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1. INTRODUCTION.

In [1], Banach proved that if T is a compact metric space and $C(T)$ is the Banach space of all continuous real valued functions on T, with the supremum norm, then

$$\lim_{\lambda\to 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda}$$

exists for all $g \in C(T)$ if and only if there exists a $t_0 \in T$ such that $|f(t_0)| > |f(t)|$ for all $t \in T$, $t \neq t_0$.

This theorem, however, is no longer true if T is a locally compact, non-compact, Hausdorff space; as can easily be seen by considering the Banach space ℓ_∞ of all bounded real valued sequences with the supremum norm.

In fact, if $\mathbb{N}$ is the set of positive integers equipped with the discrete topology, then $\ell_\infty = C(\mathbb{N})$, the space of all bounded continuous functions on $\mathbb{N}$. If we let $x = \{x_n\}_{n\geq 1} \in \ell_\infty$, where $x_1 = 1$ and $x_n = \frac{n-1}{n}$ for $n > 1$, then x peaks at $n = 1$, but because of the behaviour of x at infinity and the existence of Banach limits, it is possible to find two distinct support functionals to the ball in ℓ_∞ at x , so that x is not a smooth point.

In this note, we characterize the points of Gateaux and Fréchet differentiability of the norm function in $C(T, E)$, the space of all bounded continuous E-valued functions on the locally compact Hausdorff space T , where E is a real Banach space.

Two applications of these results are given. The first is to the space $\ell_\infty(E)$ of all bounded sequences in E , and the second to the space $B(\ell_1, E)$ of all bounded linear operators from ℓ_1 into E .

2. DEFINITIONS AND NOTATION.

In the following, E denotes a real Banach space and E^* denotes the dual of E . The unit ball of E is $B_E = \{x \in E \mid \|x\| \leq 1\}$ and its boundary $S_E = \{x \in E \mid \|x\| = 1\}$ is the unit sphere of E .

A Banach space E is said to be smooth at $x \in E \sim \{0\}$ if and only if there exists a unique hyperplane of support to B_E at $\frac{x}{\|x\|}$; that is, there exists only one continuous linear functional $\phi \in E^*$ with $\|\phi\| = 1$ such that $\phi(x) = \|x\|$. Such a linear functional $\phi \in E^*$ is called the support functional to B_E at $\frac{x}{\|x\|}$, and $\phi^{-1}(\{1\})$ is called the hyperplane of support to B_E at $\frac{x}{\|x\|}$. A Banach space E is said to be a smooth Banach space if it is smooth at every $x \in S_E$.

The norm function $\|.\| : E \to \mathbb{R}^+$ is said to be Gateaux differentiable at $x \in E \sim \{0\}$ if and only if there exists a functional $\phi \in E^*$ with

$$\lim_{\lambda \to 0} \left| \frac{\|x + \lambda h\| - \|x\|}{\lambda} - \phi(h) \right| = 0 , \qquad (*)$$

for every $h \in E$. The functional ϕ is called the Gateaux derivative of the norm at $x \in E$. The norm function $\|.\| : E \to \mathbb{R}^+$ is said to be Fréchet differentiable at $x \in E \sim \{0\}$ if and only if there exists a functional $\phi \in E^*$ such that

$$\lim_{\|h\|\to 0} \frac{| \|x + h\| - \|x\| - \phi(h) |}{\|h\|} = 0 , \qquad (**)$$

that is, the limit in (*) exists uniformly for $h \in B_E$.

It is well known, Mazur [8], that E is smooth at $x \in E \sim \{0\}$ if and only if $\lim_{\lambda\to 0} \frac{\|x + \lambda h\| - \|x\|}{\lambda}$ exists for all $h \in E$, if and only if the norm function $\|.\| : E \to \mathbb{R}^+$ is Gateaux differentiable at x.

3. SMOOTH POINTS IN $C(T, E)$.

If T is a topological space and E is a real Banach space, then $C(T, E)$ denotes the Banach space of all bounded continuous E-valued functions on T, with the supremum norm; that is,

$$C(T, E) = \{f: T \to E \mid f \text{ is bounded and continuous}\} ,$$

and $\|f\| = \sup\{\|f(t)\| : t \in T\}$, for $f \in C(T, E)$.

As mentioned earlier, Banach [1] proved that if T is a compact metric space, then $C(T) = C(T, \mathbb{R})$ is smooth at $f \neq 0$ if and only if f is a peaking function, that is, there exists a point $t_0 \in T$ such that $\|f\| = |f(t_0)| > |f(t)|$ for all $t \in T$, $t \neq t_0$.

Kondagunta [6], and Cox and Nadler [3], have characterized the points of Gateaux and Fréchet differentiability of the norm in $C(T, E)$ when T is compact Hausdorff. Cox and Nadler, in the same paper, give a characterization of the points of Fréchet differentiability of the norm in $C(T, \mathbb{R})$ when T is locally compact Hausdorff.

In this section, we generalize these results to the space $C(T, E)$ when T is locally compact Hausdorff. The techniques used by Cox and Nadler will not work in this case, since, in general, the range of $f \in C(T, E)$ is no longer relatively compact and hence no extension to an $\hat{f} \in C(\beta T, E)$ is possible. However, a slight modification of the argument given in Köthe [7], (page 352), for the corresponding result in ℓ_∞, does work.

As usual, the results on smoothness are the more difficult, and the results on Fréchet differentiability follow as a corollary.

THEOREM 3.1. Let T be a locally compact Hausdorff space and E a Banach space. A point $f \in C(T, E)$, $f \neq 0$, is a smooth point of $C(T, E)$ if and only if

(i) there exists a $t_0 \in T$ such that $\|f(t_0)\| > \|f(t)\|$ for all $t \in T$, $t \neq t_0$,

(ii) there exists a compact neighborhood K of t_0 such that

$$\sup_{t \in T \sim K} \|f(t)\| < \|f\| ,$$

(iii) $f(t_0)$ is a smooth point of E.

PROOF.

A. Assume first that the norm $\|.\| : C(T, E) \to \mathbb{R}^+$ is Gateaux differentiable at $f \in C(T, E)$, where we may (and do) assume that $\|f\| = 1$.

(I) We show first that if the mapping $\|f(.)\| : T \to \mathbb{R}^+$ achieves its maximum at $t_0 \in T$, then t_0 is unique. Suppose there exist $t_0, t_1 \in T$, $t_0 \neq t_1$, such that $\|f(t_0)\| = \|f(t_1)\| = 1$. For $t \in T$ and $\phi \in E^*$, let $\delta_{\phi,t} \in C(T, E)^*$ denote the evaluation functional given by $\delta_{\phi,t}(g) = \phi(g(t))$ for $g \in C(T, E)$; then $\|\delta_{\phi,t}\| = 1$ for all $t \in T$, $\phi \in S_{E^*}$. Using the Hahn-Banach theorem, choose $\phi_0, \phi_1 \in E^*$ with $\|\phi_0\| = \|\phi_1\| = 1$ such that $\phi_0(f(t_0)) = \phi_1(f(t_1) = 1$. Then δ_{ϕ_0,t_0} and δ_{ϕ_1,t_1} are distinct support functionals to the ball in $C(T, E)$ at f, which contradicts the fact that f is a smooth point in $C(T, E)$.

(II) We show next that given any compact set $K \subseteq T$, either $\sup_{t \in T \sim K^\circ} \|f(t)\| < 1$, or there exists a $t_0 \in T \sim K^\circ$ such that $\|f(t_0)\| = 1$.

To the contrary, suppose there exists a compact set $K \subseteq T$, such that $\sup_{t \in T \sim K^\circ} \|f(t)\| = 1$ and $\|f(t)\| < 1$, for all $t \in T \sim K^\circ$. Let $F_0(t) = \|f(t)\|$, for all $t \in T$. Then F_0 is a bounded continuous function on T and thus has an extension, F, to βT, the Stone-Čech compactification of T. Since $\sup_{t \in T \sim K^\circ} \|f(t)\| = 1$, we have $A = F^{-1}(1) \cap (\beta T \sim T) \neq \phi$. Also

$$A = \bigcap_{n=1}^{\infty} \{t \in T \sim K : F(t) > 1 - \frac{1}{n}\}$$

and thus A is a G_δ set in βT. By Čech [2], singletons in $\beta T \sim T$ are not G_δ sets, so A contains at least two distinct points p and q.

Let $\{p_\mu\}$ and $\{q_\nu\}$ be disjoint nets contained in T such that $p_\mu \to p$ and $q_\nu \to q$ in βT. For each μ, let $\phi_\mu \in E^*$, with $\|\phi_\mu\| = 1$ and $\phi_\mu(f(p_\mu)) = F(p_\mu)$. Also, for each ν, choose $\psi_\nu \in E^*$, with $\|\psi_\nu\| = 1$ and $\psi_\nu(f(q_\nu)) = F(q_\nu)$. Let $\Phi_\mu = \delta_{\phi_\mu, p_\mu}$ and $\Psi_\nu = \delta_{\psi_\nu, q_\nu}$, for each μ and ν. Then Φ_μ and $\Psi_\nu \in C(T, E)^*$ and $\|\Phi_\mu\| = \|\Psi_\nu\| = 1$, for each μ and ν.

Since the ball in $C(T, E)^*$ is w*-compact, there exist $\Phi, \Psi \in C(T, E)^*$, with $\|\Phi\| \le 1$ and $\|\Psi\| \le 1$, such that Φ is a w*-accumulation point of the net $\{\Phi_\mu\}$ and Ψ is a w*-accumulation point of the net $\{\Psi_\nu\}$. By construction, $\Phi(f) = \Psi(f) = 1$ and, thus, Φ and Ψ are support functionals to the ball in $C(T, E)$ at f. Since f is assumed to be a smooth point in $C(T, E)$, it must be that $\Phi = \Psi$. We will show that this is impossible. Let $P = \{p_\mu\} \cup \{p\}$ and $Q = \{q_\nu\} \cup \{q\}$. Then P and Q are disjoint closed subsets of βT, which is a compact Hausdorff space and therefore normal. Let $h_1, h_2 \in C(\beta T)$ with $0 \le h_1, h_2 \le 1$, $h_1 + h_2 = 1$ and $h_1(P) = h_2(Q) = 0$. Use h_i for the restriction of h_i to T, as well.

Clearly, if $g \in C(T, E)$, then $h_1 g \in \ker \Phi$ and $h_2 g \in \ker \Psi$. Since $\Phi = \Psi$, and $g \in C(T, E)$ can be written as $g = h_1 g + h_2 g$, we have $\Phi = \Psi = 0$. But, this contradicts $\|\Phi\| = \|\Psi\| = 1$. Therefore, we must have that either $\sup_{t \in T \sim K^\circ} \|f(t)\| < 1$, or there exists $t_0 \in T \sim K^\circ$ with $\|f(t_0)\| = 1$, for any compact set $K \subsetneq T$.

(III) Finally we show that (i), (ii), (iii) hold. Taking $K = \emptyset$ in (II), since $\|f\| = 1$, we see that there exists a $t_0 \in T$ with $\|f(t_0)\| = 1$; and from (I), $\|f(t_0)\| > \|f(t)\|$ for all $t \ne t_0$. Again by (II), if $K \subsetneq T$ is a compact set with $t_0 \in K^\circ$, then $\sup_{t \in T \sim K} \|f(t)\| < 1$. If there exist distinct functionals $\phi_1, \phi_2 \in E^*$ with $\|\phi_1\| = \|\phi_2\| = 1$ such that $\phi_1(f(t_0)) = \phi_2(f(t_0)) = 1$, then this implies that $\delta_{\phi_1, t_0}, \delta_{\phi_2, t_0} \in C(T, E)^*$ are distinct support functionals to the ball in $C(T, E)$ at f, which contradicts the fact that f is a smooth point. Therefore $f(t_0)$ is a smooth point of E.

B.. Conversely, suppose that $f \in C(T, E)$, $\|f\| = 1$, and (i), (ii), and (iii) hold; then there exists a unique $t_0 \in T$ such that $\|f(t_0)\| = 1$, there exists a compact set $K \subseteq T$ with $t_0 \in K^\circ$ such that $\sup_{t \in T \sim K} \|f(t)\| < 1$, and E is smooth at $f(t_0)$.

Let $g \in C(T, E)$, $g \neq 0$, and let $\delta > 0$ be such that $\|f(t)\| < \|f(t_0)\| - \delta$ for all $t \in T \sim K$. If $0 < |\lambda| < \frac{\delta}{2\|g\|}$, then for $t \in T \sim K$ we have

$$\|f(t) + \lambda g(t)\| \leq \|f(t)\| + |\lambda| \; \|g(t)\| < \|f(t_0)\| + |\lambda| \; \|g\| - \delta < \|f(t_0)\| - \frac{\delta}{2} .$$

Thus, $\|f(t) + \lambda g(t)\| < \|f(t_0)\| - \frac{\delta}{2}$ for all $t \in T \sim K$ whenever $0 < |\lambda| < \frac{\delta}{2\|g\|}$

On the other hand,

$$\|f + \lambda g\| \geq \|f\| - |\lambda| \, \|g\| = \|f(t_0)\| - |\lambda| \; \|g\| > \|f(t_0)\| - \frac{\delta}{2} \quad \text{for} \quad 0 < |\lambda| < \frac{\delta}{2\|g\|} .$$

Therefore, for $0 < |\lambda| < \frac{\delta}{2\|g\|}$,

$$\sup_{t \in T} \|f(t) + \lambda g(t)\| = \sup_{t \in K} \|f(t) + \lambda g(t)\| .$$

Since K is compact, by Kondagunta's result [6] ,

$$\lim_{\lambda \to 0} \frac{\sup_{t \in K} \|f(t) + \lambda g(t)\| - \sup_{t \in K} \|f(t)\|}{\lambda} = \lim_{\lambda \to 0} \frac{\|f \pm \lambda g\| - \|f\|}{\lambda}$$

exists for all $g \in C(T, E)$. Hence $C(T, E)$ is smooth at f .

q.e.d.

An analogous result holds for the Fréchet differentiability of the norm in $C(T, E)$.

COROLLARY 3.2. Let T be a locally compact Hausdorff space and E a Banach space. The norm function $\|.\| : C(T, E) \to \mathbb{R}^+$ is Fréchet differentiable at $f \in C(T, E)$, $f \neq 0$, if and only if

(i) there exists a unique $t_0 \in T$ such that $\|f(t_0)\| > \sup_{t \neq t_0} \|f(t)\|$;

(ii) $\{t_0\}$ is an open subset of T , that is, t_0 is an isolated point of T ;

(iii) the norm function $\|.\| : E \to \mathbb{R}^+$ is Fréchet differentiable at $f(t_0)$.

(Note: (ii) follows from (i) .)

PROOF.

A. Suppose that $\|.\| : C(T, E) \to \mathbb{R}^+$ is Fréchet differentiable at $f \in C(T, E)$, $\|f\| = 1$; then the ball in $C(T, E)$ is smooth at f , so there exists a $t_0 \in T$ and a compact neighborhood K of t_0 such that

(1) $\|f(t_0)\| > \|f(t)\|$ for all $t \in T$, $t \neq t_0$,

(2) $\sup_{t \in T \sim K} \|f(t)\| < 1$,

(3) E is smooth at $f(t_0)$.

Now

$$\lim_{\lambda \to 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda} = \lim_{\lambda \to 0} \frac{\sup_{t \in K} \|f(t) + \lambda g(t)\| - \sup_{t \in K} \|f(t)\|}{\lambda}$$

exists uniformly for $g \in B_{C(T, E)}$, and since K is compact, an appeal to the result of Cox and Nadler [3] shows that $\{t_0\}$ is open and $\|.\| : E \to \mathbb{R}^+$ is Fréchet differentiable at $f(t_0)$. Also, since $\{t_0\}$ is open and $K \sim \{t_0\}$ is compact, the uniqueness of t_0 shows that $\sup_{t \neq t_0} \|f(t)\| < 1$.

B. Conversely, suppose that (i), (ii), and (iii) hold. Using the previous theorem, we see that $\|.\| : C(T, E) \to \mathbb{R}^+$ is Gateaux differentiable at f , and taking $K = \{t_0\}$,

$$\lim_{\lambda \to 0} \frac{\|f + \lambda g\| - \|f\|}{\lambda} = \lim_{\lambda \to 0} \frac{\|f(t_0) + \lambda g(t_0)\| - \|f(t_0)\|}{\lambda}$$

exists uniformly for $g \in B_{C(T, E)}$ since $\|.\| : E \to \mathbb{R}^+$ is Fréchet differentiable at $f(t_0)$.

q.e.d.

4. APPLICATIONS.

A. $\ell_\infty(E)$. If E is a Banach space, then

$$\ell_\infty(E) = \{x = \{x_n\}_{n \geq 1} \mid x_n \in E \quad \text{for } n \geq 1 \text{ and } \sup_{n \geq 1} \|x_n\| < \infty\}$$

with the supremum norm, $\|x\| = \sup_{n \geq 1} \|x_n\|$.

THEOREM 4.1. Let E be a Banach space, then the norm function $\|.\| : \ell_\infty(E) \to \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $x = \{x_n\}_{n\geq 1}$, $x \neq 0$, if and only if

(i) there exists an n_0 such that $\|x_{n_0}\| > \|x_n\|$ for $n \neq n_0$,

(ii) $\sup_{n\neq n_0} \|x_n\| < \|x\|$,

(iii) the norm function $\|.\| : E \to \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at x_{n_0}.

PROOF. Let $\mathbb{N}$ denote the set of positive integers equipped with the discrete topology, then $\ell_\infty(E) = C(\mathbb{N}, E)$ the space of bounded continuous E-valued functions on the locally compact Hausdorff space N.

q.e.d.

B. $B(\ell_1, E)$. Let E be a Banach space, let ℓ_1 be the Banach space of all absolutely summable real valued sequences with $\|a\| = \sum_{n=1}^{\infty} |a_n|$ for $a = \{a_n\}_{n\geq 1} \in \ell_1$, and let $B(\ell_1, E)$ be the space of all bounded linear operators from ℓ_1 into E. For $n \geq 1$, let δ^n be the n^{th} basis vector in ℓ_1, that is $\delta^n = \{\delta_k^n\}_{k\geq 1}$.

THEOREM 4.2. Let E be a Banach space, then the norm function $\|.\| : B(\ell_1, E) \to \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $T \in B(\ell_1, E)$, $T \neq 0$, if and only if

(i) there exists an n_0 such that $\|T(\delta^{n_0})\| > \|T(\delta^n)\|$ for $n \neq n_0$;

(ii) $\sup_{n\neq n_0} \|T(\delta^n)\| < \|T\|$;

(iii) the norm function $\|.\| : E \to \mathbb{R}^+$ is Gateaux (Fréchet) differentiable at $T(\delta^{n_0})$.

PROOF. The mapping $\sigma : B(\ell_1, E) \to \ell_\infty(E)$ given by $\sigma(T) = \{T(\delta^n)\}_{n\geq 1}$ for $T \in B(\ell_1, E)$, is a linear isometry of $B(\ell_1, E)$ onto $\ell_\infty(E)$.

REMARKS.

1. In connection with the second example, it should be mentioned that Kheinrikh [5] has given a complete characterization of the points of Gateaux and Fréchet

differentiability of the norm in K(E, F) , the space of compact linear operators from E into F , where E and F are Banach spaces. He has also given a characterization of the points of Fréchet differentiability of the norm in B(E, F) , the space of bounded linear operators from E into F (no proofs are given in this paper). However, the more difficult question of smoothness in B(E, F) is still unanswered.

2. Regarding Theorem 3.1. perhaps this will clear up the popular misconception that, for T locally compact Hausdorff, C(T, E) (or C(T, ℝ)) is smooth at f if and only if f peaks at some $t_0 \varepsilon T$. (See e.g. Holmes [4], p. 232, #4.10). A result which is obviously false, as the example in the introduction demonstrates.

REFERENCES

1. BANACH, S. Théorie des Opérations Linéaires, Monografie Mat. 1, Warsaw, 1932.

2. CECH, E. "On bicompact spaces" Ann. of Math. 38, 823-844, 1937.

3. COX, S.H. Jr. and S.B. NADLER, Jr. "Supremum norm differentiability" Ann. Soc. Math. Polonae 15, 127-131, 1971.

4. HOLMES, R.B. Geometric Functional Analysis and its Applications, Graduate Texts in Mathematics 24, Springer-Verlag, New York, 1975.

5. KHEINRIKH, S. "The differentiability of the norm in spaces of operators" Funkcional Anal. i Prilozen 9 (4), 93-94, 1975. English translation: Functional Anal. Appl. 9 (4), 360-362, 1975.

6. KONDAGUNTA, W. "Some geometric properties of the unit cell in the space C(X, B) " Bull. Acad. Polon. Sci., Ser. Math. Astron. Phys. 19 (11), 1007-1012, 1971.

7. KÖTHE, G. Topologische linear Räume I, Die grundlehren der math. Wissenschaften 107, Springer-Verlag, Berlin, 1966. English translation: Die grundlehren der math. Wissenschaften 159, Springer-Verlag, New York, 1969.

8. MAZUR, S. "Über konvexe Mengen in linearen normierten Räumen" Studia Math. 4, 70-84, 1933.

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