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SOME ISOMETRICAL IDENTITIES IN THE WAVE EQUATION

(Dedicated to Professor Yûsaku Komatu on his 70th birthday)

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ABSTRACT. We consider the usual wave equation $u_{tt}(x,t) = c^2 u_{xx}(x,t)$ on the real line with some typical initial and boundary conditions. In each case, we establish a natural isometrical identity and inverse formula between the sourse function and the response function.

KEY WORDS AND PHRASES. *Wave equation, isometrical identity, integral transform, energy integral, inverse transform, Green's function, Hilbert space.*
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1. INTRODUCTION.

We consider the solutions of the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \text{ (c: constant, } > 0), \tag{1.1}$$

satisfying the conditions

$$\text{on the space } x \geq 0,\ u(0,t) = F(t) \text{ and } u(x,0) = 0, \tag{1.2}$$

$$\text{on the space } -\infty < x < \infty,\ u_t(x,t)|_{t=0} = F(x) \text{ and } u(x,0) = 0, \tag{1.3}$$

and

$$\text{on the space } -\infty < x < \infty,\ u(x,0) = F(x) \text{ and } u_t(x,t)|_{t=0} = 0, \tag{1.4}$$

respectively. Then, with some smoothness conditions for F, we have the integral expression for the solutions u(x,t)

$$u(x,t) = \frac{\partial}{\partial t} \int_0^t F(\xi)\, U(x, t - \xi)\, d\xi, \tag{1.5}$$

$$u(x,t) = \frac{1}{2c} \int_{-\infty}^{\infty} F(\xi)\, \theta(ct - |x - \xi|)\, d\xi, \tag{1.6}$$

and

$$u(x,t) = \frac{1}{2c} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} F(\xi)\, \theta(ct - |x - \xi|)\, d\xi, \tag{1.7}$$

respectively. See for example Inui [2], Mizohata [3] and Szmydt [6]. Here,

$$U(x,t) = \begin{cases} 1 & \text{for } |x| < ct \\ 0 & \text{for } |x| > ct \end{cases} \tag{1.8}$$

and

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0. \end{cases} \tag{1.9}$$

For (1.5) and (1.7), we have formally, for any fixed $T > 0$

$$f_T(x) = \int_0^T u(x,t)\, dt = \int_0^T F(\xi)\, U(x, T - \xi)\, d\xi \tag{1.10}$$

and

$$\int_0^T u(x,t)\, dt = \frac{1}{2c} \int_{-\infty}^{\infty} F(\xi)\, \theta(cT - |x - \xi|)\, d\xi \quad . \tag{1.11}$$

In these integral transforms (1.6), (1.10) and (1.11), we apply the general theory Saitoh [4,5] of integral transforms, and determine the Hilbert spaces formed by the images and establish natural isometrical identities and inverse formulas between F and u(x,t). As integral transforms, we can identify (1.6) and (1.11). We deal with F for $F \in L_2(-\infty, \infty)$ or $L_2(0, \infty)$, following the point of view of the general theory.

2. THE CASE WITH VARIABLE BOUNDARY VALUES WITH THE TIME.

In the case (1.10), following the general theory Saitoh [4,5], we form the reproducing kernel $K_1(x_1,x_2)$ on $[c,cT] \times [0,cT]$

$$K_1(x_1,x_2) = \int_0^T U(x_1, T - \xi)\, U(x_2, T - \xi)\, d\xi$$
$$= \min\left(T - \frac{x_1}{c}, T - \frac{x_2}{c} \right). \tag{2.1}$$

Note that $K_1(x_1,x_2)$ is the reproducing kernel for the Hilbert space H_1 composed of all the functions $f(x)$ on $[0,cT]$ satisfying that $f(x)$ are absolutely continuous on $[0,cT]$, $f'(x) \in L_2(0,cT)$, $f(cT) = 0$, and with the inner product

$$(f_1,f_2)_{H_1} = c \int_0^{cT} f_1'(x)\, f_2'(x)\, dx \quad \text{for } f_1,\ f_2 \in H_1. \tag{2.2}$$

Note that the family

$$\{ U(x , T - \xi); \quad x \in [0,cT] \} \tag{2.3}$$

is complete in $L_2(0,T)$. Hence, from the general theory we obtain

THEOREM 2.1. For any fixed $T > 0$, the images $f_T(x)$ by the integral transform (1.10) for $F \in L_2(0,T)$ form the Hilbert space H_1 admitting the reproducing kernel $K_1(x_1,x_2)$, and we obtain the isometrical identity

$$\int_0^T F(\xi)^2\, d\xi = c \int_0^{cT} f_T'(x)^2\, dx. \tag{2.4}$$

Of course, for the classical solutions $u(x,t)$ of (1.1) satisfying (1.2), we obtain an interesting isometrical identity

$$\int_0^T F(\xi)^2\, d\xi = c \int_0^{cT} \left(\frac{\partial}{\partial x} \int_0^T u(x,t) \right)^2 dx.$$

Next, we consider the inverse transform for (1.10). Note that for any $\xi \in (0,T)$, $U(x, T - \xi)$ is not absolutely continuous on $[0,cT]$. Hence, we cannot deduce the inverse formula by the general theory Saitoh [4], directly. Hence, for a reasonable family of H_1, we establish the inverse formula. We assume that $f_T \in H_1$ and further

$$f_T'(x) \text{ is absolutely continuous on } [0,cT], \quad f_T''(x) \in L_1(0,cT) \tag{2.5}$$

and

$$f_T'(0) = 0. \tag{2.6}$$

Then, we obtain, by using the reproducing property of $K_1(x,x_2)$ for H_1

$$f_T(x_2) = (f_T(x),K_1(x,x_2))_{H_1}$$

$$= c \int_0^{cT} \left\{ f_T'(x) \frac{\partial}{\partial x} \left(\int_0^T U(x,T - \xi)\, U(x_2, T - \xi)\, d\xi \right)\right\} dx$$

$$= c \left[f'_T(x) \int_0^T U(x, T - \xi)\, U(x_2, T - \xi)\, d\xi \right]_{x=0}^{cT}$$

$$- c \int_0^{cT} \left(f''_T(x) \int_0^T U(x, T - \xi)\, U(x_2, T - \xi)\, d\xi \right) dx$$

$$= \int_0^T \left(- c \int_0^{cT} f''_T(x)\, U(x, T - \xi)\, dx \right) U(x_2, T - \xi)\, d\xi . \qquad (2.7)$$

Hence, from (1.10) and the completeness of (2.3) in $L_2(0,T)$, we obtain

THEOREM 2.2. For any $f_T \in H_1$ satisfying (2.5) and (2.6), we obtain the inverse transform for (1.10)

$$F(\xi) = - c \int_0^{cT} f''_T(x)\, U(x, T - \xi)\, dx. \qquad (2.8)$$

3. THE CASE WITH INITIAL BOUNDARY VALUES.

We consider the integral transform (1.6) for functions $F \in L_2(-\infty, \infty)$. We form the reproducing kernel $K_2(x_1, x_2; t_1, t_2)$, for $x_1 \neq x_2$

$$K_2(x_1,x_2;t_1,t_2) = \frac{1}{2c} \int_{-\infty}^{\infty} \theta(ct_1 - |x_1 - \xi|)\, \theta(ct_2 - |x_2 - \xi|)\, d\xi$$

$$= \begin{cases} \dfrac{1}{2c} \left(c(t_1 + t_2) - |x_1 - x_2| \right) & \text{for } |x_1 - x_2| \leqq c(t_1 + t_2) \\ 0 & \text{for } |x_1 - x_2| \geqq c(t_1 + t_2). \end{cases} \qquad (3.1)$$

On the other hand, for any fixed $x_1 = x_2 = x$, we have

$$K_2(x_1,x_2;t_1,t_2) = K_{2,1}(t_1,t_2\,;\,x)$$

$$= \frac{1}{2c} \int_{-\infty}^{\infty} \theta(ct_1 - |x - \xi|)\, \theta(ct_2 - |x - \xi|)\, d\xi$$

$$= \min(t_1,t_2) \quad \text{for } t_1, t_2 > 0. \qquad (3.2)$$

Note that $K_{2,1}(t_1,t_2;x)$ is the reproducing kernel for the Hilbert space $H_{2,1}$ composed of all the functions $f(t)$ such that $f(t)$ are absolutely continuous on $[0,\infty)$, $f(0) = 0$, $f'(t) \in L_2(0,\infty)$, and with the inner product

$$(f_1,f_2) = \int_0^\infty f_1'(t)\, f_2'(t)\, dt \quad \text{for } f_1,\ f_2 \in H_{2,1}. \tag{3.3}$$

The family

$$\{ \theta(ct - |x - \xi|);\ t \in [0, \infty),\ \text{for fixed } x \} \tag{3.4}$$

is not complete in $L_2(-\infty, \infty)$. Indeed,

$$\int_{-\infty}^{\infty} F(\xi)\, \theta(ct - |x - \xi|)\, d\xi = 0 \ \text{for all } t > 0; \tag{3.5}$$

that is,

$$\int_{x - ct}^{x + ct} F(\xi)\, d\xi = 0 \quad \text{for all } t > 0 \tag{3.6}$$

implies that

$$F(x + ct) = - F(x - ct) \quad \text{for all} \quad t > 0. \tag{3.7}$$

In general, $F \in L_2(-\infty, \infty)$ is decomposed uniquely in the form

$$F(\xi) = F_{e(x)}(\xi) + F_{o(x)}(\xi);\quad F_{e(x)},\ F_{o(x)} \in L_2(-\infty, \infty) \tag{3.8}$$

where

$$F_{e(x)}(\xi) = \frac{1}{2}(F(\xi) + F(2x - \xi)) \quad \text{and} \quad F_{o(x)}(\xi) = \frac{1}{2}(F(\xi) - F(2x -\xi)).$$

The odd part of F satisfies (3.7) and so

$$\int_{-\infty}^{\infty} F(\xi)\, \theta(ct - |x - \xi|)\, d\xi = \int_{-\infty}^{\infty} F_{e(x)}(\xi)\, \theta(ct - |x - \xi|)\, d\xi. \tag{3.9}$$

In addition,

$$\int_{-\infty}^{\infty} F_{e(x)}(\xi)\, F_{o(x)}(\xi)\, d\xi = 0. \tag{3.10}$$

We thus obtain

THEOREM 3.1. For any fixed x, the images f(t,x) by the integral transform (1.6) for functions $F \in L_2(-\infty, \infty)$ form the Hilbert space $H_{2,1}$ admitting the reproducing kernel $K_{2,1}(t_1,t_2;x)$. Further, we obtain the isometrical identity

$$\int_0^\infty \left(\frac{\partial f(t,x)}{\partial t}\right)^2 dt = \min \frac{1}{2c} \int_{-\infty}^\infty F(\xi)^2 \, d\xi, \tag{3.11}$$

where the minimum is taken over all functions F satisfying

$$f(t,x) = \frac{1}{2c} \int_{-\infty}^\infty F(\xi)\, \theta(ct - |x - \xi|)\, d\xi, \quad F \in L_2(-\infty, \infty). \tag{3.12}$$

Moreover, the minimum is attained by F^* if and only if F^* is the even part $F_{e(x)}$ of any F satisfying (3.12).

It seems that Theorem 3.1 itself has an interesting physical sense for the wave. Note that in the classical solutions u(x,t) of (1.1) satisfying (1.3), the integral

$$\frac{1}{2} \int_{-\infty}^\infty F(\xi)^2 \, d\xi$$

can be considered as the energy of the wave u(x,t), when the dentity $\rho \equiv 1$.

For $f(t,x) \in H_{2,1}$, we take F^* such that

$$\int_0^\infty f_t(t,x)^2 \, dt = \frac{1}{2c} \int_{-\infty}^\infty F^*(\xi)^2 \, d\xi \tag{3.13}$$

in (3.11). Then, we regard F^* as the inverse of f(t,x). Then, we obtain the inverse formula, as in Theorem 2.2

THEOREM 3.2. For any $f(t,x) \in H_{2,1}$ satisfying that

$$f_t(t,x) \text{ is absolutely continuous, and } f_{tt}(t,x) \in L_1(0,\infty), \tag{3.14}$$

we obtain the inverse formula for (1.6)

$$F^*(\xi) = -\int_0^\infty \frac{\partial^2 f(t,x)}{\partial t^2}\, \theta(ct - |x - \xi|)\, dt. \tag{3.15}$$

Next, for any fixed $T > 0$, we consider the integral transform (1.6). Then, we set $K_2(x_1,x_2;T,T) = K_{2,2}(x_1,x_2;T)$. It seems that the Hilbert space $H_{2,2}$ admitting the reproducing kernel $K_{2,2}(x_1,x_2;T)$ has a much more complicated structure than H_1 and $H_{2,1}$, in essence. In order to realize the norm in $H_{2,2}$, recall the identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^2 e^{-ix\xi}\,d\xi$$

$$= \begin{cases} 1-|x| & \text{for } |x| \leqq 1 \\ 0 & \text{for } |x| \geqq 1. \end{cases} \tag{3.16}$$

See for example Butzer and Nessel [1]. We thus have the expression

$$K_{2,2}(x_1,x_2;T)$$

$$= \frac{T}{2\pi}\int_{-\infty}^{\infty} \exp\left(-i\frac{x_1\xi}{2cT}\right)\overline{\exp\left(-i\frac{x_2\xi}{2cT}\right)}\, W_1(\xi)\,d\xi, \tag{3.17}$$

where

$$W_1(\xi) = \left(\frac{\sin(\xi/2)}{\xi/2}\right)^2. \tag{3.18}$$

Note that the family

$$\left\{ \exp\left(-i\frac{x\xi}{2cT}\right) ;\ x \in (-\infty,\infty) \right\} \tag{3.19}$$

is complete in the space $L_2(W_1(\xi)d\xi)$ composed of all the functions $\hat{f}$ satisfying

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, W_1(\xi)\,d\xi < \infty . \tag{3.20}$$

Hence, from the general theory, we see that any member f of $H_{2,2}$ is expressible in the form

$$f(x) = \frac{T}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\exp\left(-i\frac{x\xi}{2cT}\right) W_1(\xi)\,d\xi \tag{3.21}$$

for a uniquely determined $\hat{f} \in L_2(W_1(\xi)d\xi)$ and we have

$$\|f\|^2_{H_{2,2}} = \frac{T}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, W_1(\xi)\,d\xi . \tag{3.22}$$

Note that since $\hat{f}(\xi)W_1(\xi) \in L_1(-\infty,\infty) \cap L_2(-\infty,\infty)$, by using the inverse transform of Fourier in the framework of L^2 space, the norm $\|f\|_{H_{2,2}}$ can be realized directly in terms of f as follows:

$$\|f\|^2_{H_{2,2}}$$

$$= \frac{2\pi}{T} \int_{-\infty}^{\infty} \left| \underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} f(x) \exp\left(\frac{ix\xi}{2cT}\right) dx \right|^2 \frac{d\xi}{W_1(\xi)}. \tag{3.23}$$

Of course, the inner product $(f_1,f_2)_{H_{2,2}}$ is given by

$$(f_1,f_2)_{H_{2,2}} = \frac{2\pi}{T} \int_{-\infty}^{\infty} \left(\underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} f_1(x) \exp\left(\frac{ix\xi}{2cT}\right) dx \right)$$

$$\overline{\left(\underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} f_2(x) \exp\left(\frac{ix\xi}{2cT}\right) dx \right)} \frac{d\xi}{W_1(\xi)}. \tag{3.24}$$

Since the family

$$\{ \theta(cT - |x - \xi|);\ \ x \in (-\infty, \infty) \} \tag{3.25}$$

is complete in $L_2(-\infty, \infty)$, we obtain

THEOREM 3.3. For any fixed $T > 0$, the images $f_T(x)$ by the integral transform (1.6) for functions $F \in L_2(-\infty, \infty)$ form the Hilbert space $H_{2,2}$ admitting the reproducing kernel $K_{2,2}(x_1,x_2;T)$, and we obtain the isometrical identity

$$\frac{1}{2c} \int_{-\infty}^{\infty} F(\xi)^2 \, d\xi$$

$$= \frac{2\pi}{T} \int_{-\infty}^{\infty} \left| \underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} f_T(x) \exp\left(\frac{ix\xi}{2cT}\right) dx \right|^2 \frac{d\xi}{W_1(\xi)}. \tag{3.26}$$

For the classical solutions $u(x,t)$ for (1.1) satisfying (1.3), from the law of conservation of energy, we obtain the identity

$$\frac{1}{2} \int_{-\infty}^{\infty} F(\xi)^2 \, d\xi = \frac{1}{2} \int_{-\infty}^{\infty} (u_t(x,t)^2 + c^2 u_x(x,t)^2) dx. \tag{3.27}$$

Compare this with (3.26). This identity (3.26) implies that this energy is expressible in terms of $u(x,t)$ only for any fixed $T > 0$. Hence, it seems that (3.26) is an interesting expression of the energy and is valuable when we evaluate the energy.

Next, we consider the inverse formula for (1.6). From the reproducing property of $K_{2,2}(x,x_2;T)$ for $H_{2,2}$, we have, for any $f_T \in H_{2,2}$

$$f_T(x_2) = (f_T(x), K_{2,2}(x,x_2;T))_{H_{2,2}}$$

$$= \frac{2\pi}{T} \int_{-\infty}^{\infty} \{ \underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} f_T(\xi_1) \exp\left(\frac{ix\,\xi_1}{2cT}\right) d\xi_1 \}$$

$$\{ \underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} \left(\frac{1}{2c} \int_{-\infty}^{\infty} \theta(cT - |\xi_2 - \xi_3|)\, \theta(cT - |x_2 - \xi_3|)\, d\xi_3 \right)$$

$$\exp\left(\frac{ix\,\xi_2}{2cT}\right) d\xi_2 \} \frac{dx}{W_1(x)} . \tag{3.27}$$

Since

$$\underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} \theta(cT - |\xi_2 - \xi_3|) \exp\left(\frac{ix\,\xi_2}{2cT}\right) d\xi_2$$

$$= 2cT\, W_1(x) \exp\left(-\frac{ix\,\xi_3}{2cT}\right), \tag{3.28}$$

we have

$$f_T(x_2) = 2\pi \int_{-\infty}^{\infty} \{ \underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} f_T(\xi_1) \exp\left(\frac{ix\,\xi_1}{2cT}\right) d\xi_1 \}$$

$$\{ \int_{-\infty}^{\infty} \theta(cT - |x_2 - \xi_3|) \exp\left(-\frac{ix\,\xi_3}{2cT}\right) d\xi_3 \} dx$$

$$= \frac{1}{2c} \lim_{M\to\infty} \int_{-\infty}^{\infty} \left[4\pi c \int_{-M}^{M} \{ \underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} f_T(\xi_1) \exp\left(\frac{ix\,\xi_1}{2cT}\right) d\xi_1 \} \right.$$

$$\left. \exp\left(-\frac{ix\,\xi_3}{2cT}\right) dx \right] \theta(cT - |x_2 - \xi_3|)\, d\xi_3. \tag{3.29}$$

Hence, from the general theorem of Saitoh [4], we obtain

THEOREM 3.4. For any $f_T \in H_{2,2}$, we have the inverse formula for (1.6)

$$F(\xi) = \underset{M\to\infty}{\text{s-lim}}\ 4\pi c \int_{-M}^{M} \{ \underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} f_T(\xi_1) \exp\left(\frac{ix\,\xi_1}{2cT}\right) d\,\xi_1\} \exp\left(-\frac{ix\xi}{2cT}\right) dx \tag{3.30}$$

in the sense of the strong convergence in $L_2(-\infty, \infty)$.

4. THE INHOMOGENEOUS CASE.

We consider the inhomogeneous case; that is,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = -\rho(x,t) \tag{4.1}$$

with the conditions

$$u(x,0) = u_t(x,t)|_{t=0} = 0. \tag{4.2}$$

Then, for a suitable ρ, we obtain the integral expression of the solutions $u(x,t)$

$$u(x,t) = \frac{c}{2}\int_0^t \int_{-\infty}^{\infty} \rho(\xi,\hat{t})\ \theta(c(t-\hat{t}) - |x-\xi|)\ d\xi\ d\hat{t}. \tag{4.3}$$

First, for any fixed $T > 0$ and functions F satisfying

$$\int_0^T \int_{-\infty}^{\infty} F(\xi,t)^2\ d\xi\ dt < \infty, \tag{4.4}$$

we consider the integral transform

$$f_T(x) = \int_0^T \int_{-\infty}^{\infty} F(\xi,t)\ \theta(c(T-t) - |x-\xi|)\ d\xi\ dt. \tag{4.5}$$

We form the reproducing kernel

$$K_{3,1}(x_1,x_2;T) = \int_0^T \int_{-\infty}^{\infty} \theta(c(T-t) - |x_1-\xi|)\theta(c(T-t) - |x_2-\xi|)\ d\xi\ dt$$

$$= \begin{cases} cT^2 \left(1 - \dfrac{|x_1 - x_2|}{2cT} \right)^2 & \text{for } |x_1 - x_2| \leqq 2cT \\ 0 & \text{for } |x_1 - x_2| \geqq 2cT . \end{cases} \tag{4.6}$$

From (3.6), we have the expression

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi} W_2(\xi)\, d\xi = \begin{cases} (1 - |x|)^2 & \text{for } |x| \leqq 1 \\ 0 & \text{for } |x| \geqq 1, \end{cases} \tag{4.7}$$

where

$$W_2(\xi) = \int_{-\infty}^{\infty} W_1(x)\, W_1(\xi - x)\, dx, \text{ with (3.18).} \tag{4.8}$$

We thus obtain the expression

$$K_{3,1}(x_1, x_2; T)$$

$$= cT^2 \int_{-\infty}^{\infty} \exp\left(-i \frac{x_1 \xi}{2cT} \right) \overline{\exp\left(-i \frac{x_2 \xi}{2cT} \right)}\, W_2(\xi)\, d\xi . \tag{4.9}$$

Hence, we obtain, as in Theorem 3.2

THEOREM 4.1. For any fixed $T > 0$, the images $f_T(x)$ by the integral transform (4.5) for functions F satisfying (4.4) form the Hilbert space $H_3(T)$ composed of all functions f_T on $(-\infty, \infty)$ with finite norms

$$||f_T||^2_{H_3(T)}$$

$$= \frac{1}{2\pi cT^2} \int_{-\infty}^{\infty} \left| \underset{L\to\infty}{\text{l.i.m.}} \int_{-L}^{L} f_T(x) \exp\left(\frac{ix\xi}{2cT} \right) dx \right|^2 \frac{d\xi}{W_2(\xi)}, \tag{4.10}$$

and admitting the reproducing kernel $K_{3,1}(x_1, x_2; T)$.

Furthermore, we obtain the isometrical identity

$$||f_T||^2_{H_3(T)} = \int_0^T \int_{-\infty}^{\infty} F(\xi, t)^2\, d\xi\, dt. \tag{4.11}$$

As in Theorem 3.4, we obtain

THEOREM 4.2. For any fixed $T > 0$ and $f_T \in H_3(T)$, we obtain the inverse formula for (4.5)

$$F(\xi, t) = \frac{2}{\pi T} \operatorname*{s-lim}_{M\to\infty} \int_{-M}^{M} \left\{ \frac{1}{x} \sin \frac{x(T-t)}{2T} \exp\left(-\frac{ix\xi}{2cT}\right)\right\}$$

$$\left\{ \operatorname*{l.i.m.}_{L\to\infty} \int_{-L}^{L} f_T(\xi_1) \exp\left(\frac{i\,\xi_1\, x}{2cT}\right) d\xi_1 \right\} \frac{dx}{W_2(x)} . \tag{4.12}$$

Next, for any fixed x, we consider the integral transform (4.3) for functions $\rho(\xi,t)$ satisfying

$$\int_0^\infty \int_{-\infty}^\infty \rho(\xi,t)^2 \, d\xi \, dt < \infty . \tag{4.13}$$

We form the reproducing kernel

$$K_{3,2}(t_1,t_2;x) = \frac{c}{2} \int_0^\infty \int_{-\infty}^\infty \theta\,(c(t_1 - \hat{t}) - |x-\xi|\,)\, \chi(\hat{t};\,(0,t_1))$$

$$\theta(c(t_2 - \hat{t}) - |x-\xi|)\ \ \chi(\hat{t};\,(0,t_2))\, d\xi\, d\hat{t}, \tag{4.14}$$

where

$$\chi(\hat{t};(0,t)) = \begin{cases} 1 & \text{for } 0 < \hat{t} < t \\ 0 & \text{for } \hat{t} \leqq 0 ,\quad t \leqq \hat{t} . \end{cases}$$

Then, we have

$$K_{3,2}(t_1,t_2;x) = \frac{c^2}{2} \min\{ t_1^2 , t_2^2 \} \quad \text{for } t_1,t_2 \geqq 0. \tag{4.15}$$

Note that $K_{3,2}(t_1,t_2;x)$ is the reproducing kernel for the Hilbert space $H_3(x)$ composed of all functions $f(t)$ such that $f(t)$ are absolutely continuous on $[0,\infty)$ and $f(0) = 0$, and with finite norms

$$\left\{ \frac{1}{c^2} \int_0^\infty f'(t)^2 \frac{dt}{t} \right\}^{\frac{1}{2}} < \infty . \tag{4.16}$$

Since, the family

$$\{\theta(c(t - \hat{t}) - |x-\xi|)\ \chi(\hat{t};(0,t);\ t \in (0,\infty) \}$$

is not complete in $L_2((-\infty, \infty)\times (0, \infty))$, we obtain

THEOREM 4.3. For any fixed x, the images $u(x,t)$ by the integral transform (4.3) for functions ρ satisfying (4.13) form the Hilbert space $H_3(x)$ composed of all functions $f(t)$ on $(0,\infty)$ with finite norms (4.16) and admitting the reproducing kernel $K_{3,2}(t_1,t_2;x)$.

Furthermore, we obtain the isometrical identity

$$\frac{1}{c^2}\int_0^\infty \left(\frac{\partial u(x,t)}{\partial t}\right)^2 \frac{dt}{t}$$

$$= \min \frac{c}{2}\int_0^\infty\int_{-\infty}^\infty \rho(\xi,\hat{t})^2 \, d\xi \, d\hat{t}$$

$$:= \frac{c}{2}\int_0^\infty\int_{-\infty}^\infty \rho^*(\xi,\hat{t})^2 \, d\xi \, d\hat{t}, \tag{4.17}$$

where the minimum is taken over all ρ satisfying (4.3).

Further, we obtain, as in Theorem 2.2

THEOREM 4.4. For any $u(x,t) \in H_3(x)$ satisfying

$$\int_0^\infty \left|\frac{\partial^2 u(x,t)}{\partial t^2}\right| \frac{dt}{t} < \infty, \tag{4.18}$$

we obtain the inverse formula for (4.3)

$$\rho^*(\xi,\hat{t})$$

$$= -\frac{1}{c^2}\int_{\hat{t}}^\infty \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial u(x,t)}{\partial t}\right) \theta(c(t-\hat{t}) - |x-\xi|)\frac{dt}{t}. \tag{4.19}$$

The Green's functions $G_2(x,y;x', y';t)$ and $G_3(x,y,z;x',y',z';t)$ for $R^2 = (x,y)$ and $R^3 = (x,y,z)$ for the corresponding wave equations are given by

$$G_2(x,y;x',y';t)$$

$$= \frac{c\,\theta(ct - \sqrt{(x-x')^2 + (y-y')^2})}{2\pi\sqrt{c^2t^2 - [(x-x')^2 + (y-y')^2]}} \tag{4.20}$$

and

$$G_3(x,y,z;x',y',z';t)$$

$$= \frac{\delta\left(t - \sqrt{\dfrac{(x-x')^2 + (y-y')^2 + (z-z')^2}{c}}\right)}{4\pi\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}. \tag{4.21}$$

Hence, we note that the arguments of this paper in the one dimensional space do not valid directly in the cases R^2 and R^3 for the sigularities of the Green's functions (4.20) and (4.21).

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