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DUAL INTEGRAL EQUATIONS WITH FOX'S H-FUNCTION KERNEL

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ABSTRACT. The dual integral equations involving Bessel function kernels were first considered by Weber in 1873. The problem comprised of finding potential of an electrified disc which belongs to a general category of mixed boundary value problems. Titchmarsh gave the formal solution using Wiener-Hopf procedure. We use this direct method as improvised by Busbridge to solve a class of dual integral equations which can be reduced to other known kernels by particularizing the parameters in the Fox's H-function.

KEY WORDS AND PHRASES. DUAL INTEGRAL EQUATION, Fox's H-function, Wiener-Hopf technique.
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1. INTRODUCTION

Most of the dual integral equations which we meet in the solution of mixed boundary value problems can be typified by

$$\int_0^\infty w(u)A(u)K(u,x)\,du = f(x) \ , \qquad x \in I_1 \ , \tag{1.1}$$

$$\int_0^\infty A(u)K(u,x)\,du = g(x) \ , \qquad x \in I_2 \ , \tag{1.2}$$

where $w(u)$ is a function of u alone and is called the 'weight function' $K(u,x)$ is the kernel of this pair of equations, and $A(u)$ is to be found. $I_1 = \{x: 0 \le x < 1\}$, $I_2 = \{x: x > 1\}$. Most recent literature on dual integral equations has been incorporated in a recent book by Sneddon [1].

About two decades ago, Johnson [2] investigated the method of solution of Titchmarsh [6] and found that the $K(u,x)$ can be the G-function with Titchmarsh's method applicable. Kesarwani [4, 5], taking cue from Buschman [1], solved the dual integral equations with G-functions as kernels. Saxena [7, 8] found the solution of (1.1) and (1.2) with $w(u) = 1$, had taken $K(u,x)$ to be a H-function and used fractional integral operators.

We have closely followed Titchmarsh's method in solving (1.1) and (1.2) with $g(x) = 0$, $w(u) = u$, and $K(u,x)$ a H-function. Our solution, therefore, differs from others cited above in view of the method adopted and, hence, is of interest in itself.

2. THE SOLUTION.

We shall find the formal solution of the dual integral equations given by

$$\int_0^\infty f(y)y\, H^{m,n}_{p,q}\left[xy \;\middle|\; \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] dy = e(x) \qquad (0 < x < 1) \tag{2.1}$$

$$\int_0^\infty f(y)\, H^{m,n}_{p,q}\left[xy \;\middle|\; \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] dy = 0 \qquad (x > 1) \tag{2.2}$$

where m,n,p and q are integers with $1 \le m \le q$, $0 \le n \le p$, $p < q$, and $\theta = 0$, where

$$\theta = \sum_1^n \alpha_i - \sum_{n+1}^p \alpha_i + \sum_1^m \beta_i - \sum_{m+1}^q \beta_i \tag{2.3}$$

and

$$H^{m,n}_{p,q}\left[x \;\middle|\; \begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\prod_{j=1}^m \Gamma(b_j-\beta_j s)\prod_{j=1}^n \Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^p \Gamma(a_j-\alpha_j s)}\, x^s\, ds \tag{2.4}$$

where $\mathrm{Re}\,\frac{b_j}{\beta_j} > \gamma > \mathrm{Re}\,\frac{a_k-1}{\alpha_k}$, $j=1,\dots,m$; $k=1,\dots,n$.

We apply the formal manipulations given by Titchmarsh to (2.1) and (2.2). If Parseval's formula is applied to the left-hand sides of (2.1) and (2.2), the results are

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)\, \frac{\prod_{j=1}^m \Gamma(b_j+\beta_j+\alpha\beta_j-\beta_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j-\beta_j-\alpha\beta_j+\beta_j s)} \cdot$$

$$\cdot \frac{\prod_{j=1}^n \Gamma(1-a_j-\alpha_j-\alpha\alpha_j+\alpha_j s)}{\prod_{j=1}^n \Gamma(a_j+\alpha_j+\alpha\alpha_j-\alpha_j s)}\, x^{s-\alpha-1}\, ds = e(x) \qquad (0 < x < 1) \tag{2.5}$$

and

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)\, \frac{\prod_{j=1}^m \Gamma(b_j+\beta_j-\beta_j s)}{\prod_{j=m+1}^q \Gamma(1-b_j-\beta_j+\beta_j s)} \cdot$$

$$\cdot \frac{\prod_{j=1}^n \Gamma(1-a_j-\alpha_j+\alpha_j s)}{\prod_{j=1}^n \Gamma(a_j+\alpha_j-\alpha_j s)}\, x^{s-1}\, ds = 0\,, \qquad (x > 1)\,. \tag{2.6}$$

The substitution of

$$F(s)=\frac{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j+\alpha\alpha_j-\alpha_j s)}{\prod_{j=1}^{m}\Gamma(b_j+\alpha\beta_j+\beta_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)}\,Y(s) \tag{2.7}$$

into the last two equations gives

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} Y(s)\,\frac{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\alpha\beta_j-\beta_j+\beta_j s)}\,\cdot$$

$$\cdot\,\frac{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j-\alpha\alpha_j+\alpha_j s)}{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)}\,x^{s-\alpha-1}\,ds = e(x)\quad,\qquad (0<x<1)\quad, \tag{2.8}$$

and

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} Y(s)\,\frac{\prod_{j=1}^{n}\Gamma(\beta_j+b_j-\beta_j s)}{\prod_{j=1}^{m}\Gamma(b_j+\alpha\beta_j+\beta_j-\beta_j s)}\,\cdot$$

$$\cdot\,\frac{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j+\alpha\alpha_j-\alpha_j s)}{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j-\alpha_j s)}\,x^{s-1}\,ds = 0\quad,\qquad (x>1)\quad, \tag{2.9}$$

Multiplying (2.8) by $x^{\alpha-w}$, where $\mathrm{Re}(s-w)>0$ and integrating over $(0,1)$, we obtain

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\alpha\beta_j-\beta_j+\beta_j s)}\,\cdot$$

$$\cdot\,\frac{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j-\alpha\alpha_j+\alpha_j s)}{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)}\,\frac{Y(s)}{s-w}\,ds = \int_0^1 e(x)x^{\alpha-w}\,dx$$

$$= E(\alpha-w+1)\qquad\qquad (\mathrm{Re}\ w<k)\ . \tag{2.10}$$

Moving the line of integration from $\mathrm{Re}\ s=k$ to $\mathrm{Re}\ s=k'<\mathrm{Re}\ w$ and assuming $\mathrm{Re}\,\frac{(b_j+\beta_j-1)}{\beta_j}<k'$ $(j=m+1,\dots,q)$ and $\mathrm{Re}\,\frac{(a_j+\alpha_j+\alpha\alpha_j-1)}{\alpha_j}<k'$ $(j=1,\dots,n)$,

we obtain

$$\frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty}\frac{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j-\alpha\alpha_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\alpha\beta_j-\beta_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)}\frac{Y(s)}{s-w}\,ds$$

$$= E(1+\alpha-w) - \frac{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j w)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j-\alpha\alpha_j+\alpha_j w)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\alpha\beta_j-\beta_j+\beta_j w)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j w)}\,Y(w) \qquad (2.11)$$

The integral occurring on the left-hand side of this equation is a regular function of w for $\text{Re } w > k'$. Therefore, so is the function on the right-hand side. Hence so also is

$$Y(w) - \frac{\prod_{j=m+1}^{q}\Gamma(1-b_j-\alpha\beta_j-\beta_j+\beta_j w)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j w)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j w)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j-\alpha\alpha_j+\alpha_j w)}\,E(1+\alpha-w)$$

If we assume suitable conditions at infinity, we have

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\left\{Y(s) - \frac{\prod_{j=m+1}^{q}\Gamma(1-b_j-\alpha\beta_j-\beta_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j-\alpha\alpha_j+\alpha_j s)}\,E(1+\alpha-s)\right\}\frac{ds}{s-w} = 0$$

$$(\text{Re } w < k) \qquad (2.12)$$

Similarly, multiplying (2.9) by ρ^{-w}, $\text{Re}(s-w) < 0$, and integrating over $(1,\infty)$, we obtain

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\prod_{j=1}^{m}\Gamma(\beta_j+b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j+\alpha\alpha_j-\alpha_j s)}{\prod_{j=1}^{m}\Gamma(b_j+\beta_j+\alpha\beta_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j-\alpha_j s)}\frac{Y(s)}{s-w}\,ds = 0, \quad (\text{Re } w > k'), \quad (2.13)$$

We conclude as before that

$$\frac{\prod_{j=1}^{m}\Gamma(\beta_j+b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j+\alpha\alpha_j-\alpha_j s)}{\prod_{j=1}^{m}\Gamma(b_j+\beta_j+\alpha\beta_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j-\alpha_j s)}\,Y(s) \quad , \qquad (2.14)$$

and so Y is regular for Re s < k . Hence

$$\frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty}\frac{Y(s)}{s-w}\,ds = 0 \qquad (\text{Re } w > k') \tag{2.15}$$

Moving the line of integration from Re s=k' to Re s=k , we have

$$\frac{1}{2\pi i}\int_{k'-i\infty}^{k'+i\infty}\frac{Y(s)}{s-w}\,ds = Y(w) \qquad (\text{Re } w < k) \tag{2.16}$$

It follows from (2.12) and (2.15) that

$$Y(s) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\prod_{j=m+1}^{p}\Gamma(1-b_j-\alpha\beta_j-\beta_j+\beta_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)}\cdot\frac{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)}{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha\alpha_j+\alpha_j s)}\cdot\frac{E(1+\alpha-s)}{s-w}\,ds \tag{2.17}$$

If Mellin's inversion formula is applied to (2.7) , then

$$f(y) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j+\beta_j s)\ \prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j+\alpha\alpha_j-\alpha_j s)}{\prod_{j=1}^{m}\Gamma(b_j+\alpha\beta_j+\beta_j-\beta_j s)\ \prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j+\alpha_j s)}\,Y(s)y^{-s}\,ds \tag{2.18}$$

Equations (2.17) and (2.18) give a solution to (2.1) and (2.2) .

The cases of Meijer's G-function and lower transcendents follow in a perspicuous manner on particularizing the parameter in the H-function.

The identity

$$x^{\gamma}\, H^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right] = H^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}(a_p+\gamma\alpha_p,\alpha_p)\\(b_q+\gamma\beta_q,\beta_q)\end{matrix}\right]$$

allows the absorption of any power of x in the H-function. Therefore, we could have multiplied the kernels of (2.1) by x^{α}, put $x^{\alpha}\,e(x) = g(x)$ and solved the pair thus obtained without any loss of generality.

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