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SOME TORSION THEORETICAL CHARACTERIZATIONS OF RINGS

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ABSTRACT. Rings whose torsion free modules are quasi-injective, quasi-projective or coflat have been characterized in the context of certain torsion theories.

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1. INTRODUCTION AND BACKGROUND.

Throughout we shall assume that rings are associative, have the identity element, and modules are left unital. R will denote a ring with identity, R-Mod the category of left R-modules, and E(M) the injective hull of a left R-module M. Also, for a left R-module M, $A \subseteq' M$ will denote that A is an essential submodule of M, and Z(M) its singular submodule (which consists of those elements whose annihilators are essential left ideals of R). M is called singular if Z(M) = M, and non-singular in case Z(M) = 0. For fundamental definitions and results concerning torsion theories, we refer to [1]. Recall that a pair (G, F) of classes of left R-modules is called Goldie torsion theory if G is the smallest torsion class containing all modules B/A, where $A \subseteq' B$, and the torsion free class F is precisely the class of non-singular modules. For an R-module M, T(M) denotes its torsion submodule. For a fixed left R-module M, an R-module Q is called M-injective (see [2]) if for each short exact sequence $0 \to N \to M \to M/N \to 0$ the sequence $0 \to Hom_R(M/N, Q) \to Hom_R(M, Q) \to Hom_R(N, Q) \to 0$ is exact. M-projective modules can be defined dually. An R-module M is quasi-injective (quasi-projective) if and only if it is M-injective (M-projective). A left R-module M is called coflat if given $f \in Hom_R(I, M)$, when I is a finitely generated left ideal of R, $\exists\ g \in Hom_R(R, M)$, such that $g|_I = f$ (see [3], [4] and [5, Prop. 1.6, p. 351].

2. RESULTS.

We start with the following.

THEOREM 1. Let (G, F) be the Goldie torsion theory for R-Mod. Then each torsion free R-module is quasi-injective if and only if $\bar{R} = R/T(R)$ is semisimple artinian.

PROOF. Suppose each torsion free R-module is quasi-injective. It is clear that the hypothesis carries over to $\bar{R} = R/T(R)$. Let ${}_{\bar{R}}M$ be a torsion free module in the induced torsion theory for $\bar{R}$-Mod. Since ${}_{\bar{R}}M \oplus E({}_{\bar{R}}\bar{R})$ is torsion free, it follows that ${}_{\bar{R}}M \oplus E({}_{\bar{R}}\bar{R})$ is ($\bar{R}$-) quasi-injective. So, ${}_{\bar{R}}M$ is $E(\bar{R})$-injective. Hence ${}_{\bar{R}}M$ is ($\bar{R}$-) injective. Hence by [6, Theorem 3.1], $\bar{R}$ is semisimple artinian. Conversely, suppose $\bar{R}$ is semisimple artinian. Let ${}_{R}M$ be a torsion free module. Then M is a (torsion free) $\bar{R}$-module. Hence ${}_{\bar{R}}M$ is ($\bar{R}$-) quasi-injective. This implies that M is quasi-injective, as an R-module.

Now we examine the structure of rings whose torsion free modules are quasi-projective. The following lemmas are needed.

LEMMA 1. If ${}_{R}R$ is π-quasi-projective (i.e., direct product of any copies of R is quasi-projective), then each projective module is π-projective (see Fuller and Hill [7, Cor. 2.2, p. 370]).

LEMMA 2. Let R be any ring. Then each direct product of ${}_{R}R$ is projective if and only if R is left perfect and right coherent (see Goodearl [8, Theorem 5.15, p. 144]).

LEMMA 3. Let M be a faithful R-module. Then every M-projective R-module having a projective cover is projective (see [2, Theorem 9]).

THEOREM 2. Let (G, F) be the Goldie torsion theory for R-Mod, and assume that the maximal left quotient ring of $\bar{R} = R/T(R)$ is semisimple artinian and ($\bar{R}$-) flat. Then each torsion free left R-module is quasi-projective if and only if $\bar{R} = R/T(R)$ is left perfect and left hereditary.

PROOF. Suppose each torsion free left R-module is quasi-projective. Then the hypothesis carries over to $\bar{R}$. Since $\bar{R}$ is torsion free in the induced torsion theory for $\bar{R}$-Mod, it follows that the direct product, $\bar{R}^A$, of card A copies of $\bar{R}$ is torsion free, for each set A. So $\bar{R}^A$ is ($\bar{R}$-) quasi-projective for each set A. Hence by Lemma 2, $\bar{R}$ is π-projective and by Lemma 3, $\bar{R}$ is left perfect. Now let $\bar{I}$ be a left ideal of $\bar{R}$. Then $\bar{I} \oplus \bar{R}$ is torsion free. So $\bar{I} \oplus \bar{R}$ is ($\bar{R}$-) quasi-projective. This means that $\bar{I}$ is ($\bar{R}$-) projective. Since $\bar{R}$ is left perfect, $\bar{I}$ has a projective cover. Hence $\bar{I}$ is projective in the usual sense by Lemma 3. Thus $\bar{R}$ is left hereditary. Conversely, suppose $\bar{R} = R/T(R)$ is left perfect and left hereditary. Since the maximal left quotient ring of $\bar{R}$ is semisimple artinian by the hypothesis and $Z({}_{\bar{R}}\bar{R}) = 0$, it follows from Sandomierski ([9], Theorem 1.6, p. 115) that $\bar{R}$ is finite dimensional. Let M be a torsion free R-module. Then M is a torsion free R/T(R)-module. Hence by Cheatham and Enochs [10, Theorem 1], M is flat as R/T(R)-module. Since R/T(R) is left perfect, M is projective as R/T(R)-module by a well-known theorem of Bass (see [11, p. 315]). This implies that M is quasi-projective as an R-module.

COROLLARY. Let R be an integral domain. Then each torsion free module (in the classical sense) is quasi-projective if and only if R is a field.

Next, we characterize rings all of whose torsion free modules are coflat. The following lemmas are needed for this purpose.

LEMMA 4 (R. Ware [12, Lemma 2.2, p. 238]). Let R be a ring and $0 \to K \to P \to M \to 0$ be an exact sequence of R-modules with P projective. Then the following are equivalent:

(1) M is flat.

(2) Given any $x \in K$, $\exists$ a homomorphism $g: P \to K$, such that $xg = x$.

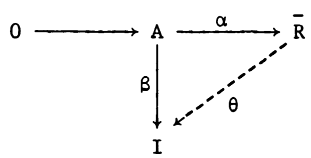
(3) Given any $x_1, \ldots, x_k \in K$, $\exists$ a homomorphism $g: P \to K$, such that $x_i g = x_i$, for $i = 1, \ldots, k$.

LEMMA 5. Let (T, F) be any hereditary torsion theory for R-Mod and $\bar{R} = R/T(R)$. Then a torsion free left R-module M is (R-) coflat if and only if M is ($\bar{R}$-) coflat.

PROOF. Suppose M is coflat as an $\bar{R}$ (= $R/T(R)$)-module. We show that M is coflat as an R-module. Let I be a finitely generated left ideal of R, and let $f: I \to M$ be an (R-) homomorphism. Let us define the map $g: (I + T(R))/T(R) \to M$ by $g(x + T(R)) = f(x)$, $x \in I$. This map is well-defined as an R-homomorphism, since M is torsion free. Also, it is clear that g is an $R/T(R)$-homomorphism. Since M is $R/T(R)$-coflat, $\exists$ an $m \in M$ such that $g(x + T(R)) = m(x + T(R))$, $\forall (x+T(R)) \in (I + T(R))/T(R)$. Thus $f(x) = mx$, $\forall x \in I$. Hence M is (R-) coflat. By a similar argument, it can be shown that if M is (R-) coflat then M is ($\bar{R}$-) coflat.

THEOREM 3. Let (T, F) be a hereditary torsion theory for R-Mod. Then each torsion free left R-module is coflat if and only if $\bar{R} = R/T(R)$ is (von Neumann) regular.

PROOF. Suppose each torsion free module is coflat. Let (T', F') be the induced torsion theory for $\bar{R}$-Mod. Let ${}_{\bar{R}}M \in F'$. Then ${}_R M \in F$. Hence, by the assumption, ${}_R M$ is coflat and, consequently, M is ($\bar{R}$-) coflat. Now let I be any left ideal of $\bar{R}$. Since $I \in F'$, I is ($\bar{R}$-) coflat. Let $x \in {}_{\bar{R}}I$, and let A be the principal ideal generated by x. In other words, ${}_{\bar{R}}A$ is a principal subideal of ${}_{\bar{R}}I$. Let us now consider the diagram:



where α and β are inclusion maps. Since I is coflat, $\exists$ an ($\bar{R}$-) homomorphism $\theta: \bar{R} \to I$ such that $\theta\alpha = \beta$. Thus $\theta|_A = \text{id}$. Now consider the sequence $0 \to I \to \bar{R} \to \bar{R}/I \to 0$. Then, as shown above, $\exists$ an ($\bar{R}$-) homomorphism $\theta: \bar{R} \to I$ such that $\theta(x) = x$, $\forall x \in I$. Since ${}_{\bar{R}}\bar{R}$ is projective, it follows from Lemma 4, that $\bar{R}/I$ is flat. Thus every cyclic left $\bar{R}$-module is flat. Hence $\bar{R}$ is regular (in the sense of von Neumann). Conversely, suppose $\bar{R}$ is regular. Let M be a torsion free

(R-) module. Then, it is easy to see that M is (torsion free) $\bar{R}$-module. Hence M is coflat as an $\bar{R}$-module by Damiano ([5, Prop. 1.11, p. 353]). Now it follows from Lemma 5 that M is coflat as an R-module.

Since a regular ring without zero divisors is a division ring, the following corollary is immediate.

COROLLARY. Let R be an integral domain. Then each torsion free module (in the classical sense) is coflat if and only if R is a field.

In [13], Megibben proved that R is left semihereditary if and only if each factor module of an absolutely pure module is absolutely pure. This characterization remains valid if absolutely pure is replaced by coflat. We shall now obtain a characterization of left semihereditary rings in terms of a condition imposed on their (Goldie) torsion free coflat modules. First we state two lemmas.

LEMMA 6. Let M_0 be an R-module which is either a homomorphic image of $_RM$ or a submodule of $_RM$. Then every M-projective (M-injective) module is M_0-projective (M_0-injective) (see [2, Prop. 3]).

LEMMA 7. Let R be any ring. Then each finitely generated 'R-projective' (in the sense of Azumaya [2]) is a projective R-module (see [2, Prop. 4]).

THEOREM 4. Let (G, F) be the Goldie torsion theory for R-Mod. Then the following are equivalent:

(1) R is left semihereditary.

(2) $Z(R) = 0$, and each homomorphic image of a torsion free coflat module is coflat.

PROOF. In the light of the aforementioned remarks, the proof will be complete if we show that (2) $\Longrightarrow$ (1). So, let us assume (2). Since $Z(R) = 0$, $_RR \in F$. Hence $E(R) \in F$, and consequently, each homomorphic image of $E(R)$ is coflat. Now let I be a finitely generated left ideal of R. Consider the diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{\beta} & R \\ & & \downarrow{\scriptstyle\alpha} & & \\ E & \xrightarrow{\pi} & K & \longrightarrow & 0 \end{array}$$

in which $E = E(R)$ and K denotes a factor module of E. By the assumption, K is coflat. Hence $\exists$ a homomorphism $\mu: R \to K$, such that $\mu\beta = \alpha$. Since $_RR$ is projective, $\exists$ a homomorphism $\theta: R \to E$, such that $\pi\theta = \mu$. Let $\phi = \theta\beta$. Then $\phi: I \to E$ is an (R-) homomorphism such that $\pi\phi = \pi\theta\beta = \mu\beta = \alpha$. Thus I is E-projective. Hence by Lemma 6, I is 'R-projective.' Since I is finitely generated, it follows from Lemma 7 that I is projective (in the usual sense). Hence R is left semihereditary.

COROLLARY 1. Let R be an integral domain. Then R is a Prüfer domain if and only if each homomorphic image of a torsion free (in the classical sense) coflat module is coflat. Furthermore, if R is noetherian then R is a Dedekind domain if and only if each homomorphic image of a torsion free injective module is injective.

COROLLARY 2. Let R be noetherian and (G, F) be the Goldie torsion theory for R-Mod. Then R is left hereditary if and only if $Z(R) = 0$, and each homomorphic image of a torsion free injective module is injective.

Concerning semihereditary rings, we also note the following.

THEOREM 5. Let (T, F) be a faithful (i.e., $T(R) = 0$) hereditary torsion theory for R-Mod. Assume that Q, the ring of left quotients of R with respect to T, is semisimple artinian and flat as a left R-module. Then the following are equivalent:

(1) Each finitely generated submodule of a finitely generated torsion free module is quasi-projective.

(2) R is left semihereditary.

PROOF. 1) (1) ⇒ (2). Let I be a finitely generated left ideal of R. Then $I \oplus R$ is quasi-projective. This implies that I is 'R-projective.' Hence by Lemma 7, I is projective. Therefore, R is left semihereditary.

2) (2) ⇒ (1). Let $_RM$ be a finitely generated torsion free module. Then $_RM$ is projective by Theorem 2.2 on p. 140 in [14]. Since R is left semihereditary, it follows from a well-known property of these rings (see, for example [8, p. 10]), that each finitely generated submodule of M is (in fact) projective.

COROLLARY 1. For each finitely generated R-module M, T(M) is a summand of M.

PROOF. Since $M/T(M) \oplus {}_RR$ is finitely generated and torsion free, it is quasi-projective. So, M/T(M) is 'R-projective.' Hence by Lemma 7, M/T(M) is projective. Thus the sequence $0 \to T(M) \to M \to M/T(M) \to 0$, splits. Hence T(M) is a summand of M.

COROLLARY 2. T is stable (i.e., closed under injective hulls).

PROOF. Using an argument of Teply, we first show that $T \supseteq G$, where G denotes the Goldie torsion class. Suppose ∃ an essential left ideal I of R such that $R/I \in G$ but $R/I \notin T$. Let $T(R/I) = K/I \neq R/I$. Since $(R/I)/(K/I) \cong R/K$, $R/K \in F$. Now, as $R \oplus R/K$ is both torsion free and finitely generated, it follows that $R \oplus R/K$ is quasi-projective. Hence, as before, R/K is projective. This implies that K is a summand of R, i.e., there exists a left ideal L of R, such that $R = L \oplus K$. But $L \cap I \subseteq L \cap K = (0)$ and $I \subseteq' R$. This is a contradiction. Hence $R/I \in T$. Now let $B \in T$. Then $E(B)/B \in G$. Hence $E(B)/B \in T$. Since the sequence $0 \to B \to E(B) \to E(B)/B \to 0$ is exact and since B and E(B)/B are in T, it follows that $E(B) \in T$. Thus T is stable.

We now recall the well-known fact that R is semisimple artinian if each member of R-Mod is injective. A subsequent result of Osofsky [15] states that R is semisimple artinian if and only if each cyclic left R-module is injective. In [6], Alin and Dickson have characterized rings all of whose torsion free modules are injective in the context of Goldie torsion theory. In fact, as expressed in the proof of Theorem 1 of this note, if R is a ring with zero singular ideal then each (Goldie) torsion free module is injective if and only if R is semisimple artinian ([6], Theorem 3.1). However, as indicated by the following remark, this need not be the case if one assumes that only cyclic torsion free modules are injective.

REMARK. Let R be a ring with $Z(R) = 0$ and (G, F) be the Goldie torsion theory for R-Mod. If each torsion free cyclic module is injective, then every nonzero torsion free module contains a nonzero projective submodule. For, if K is a nonzero torsion free cyclic module, then $K \cong R/A$, where A is a left ideal of R. Since $R/A \in F$, i.e., $Z(R/A) = 0$, A is closed in R. On the other hand, since each torsion free cyclic left R-module is injective, R is left self-injective. This implies that $\exists$ an idempotent $e(\neq 1)$ in R, such that $A \subseteq' Re$. But A is closed in R, so $A = Re$. Hence the sequence $0 \to A \to R \to R/A \to 0$ splits. So $K(\cong R/A)$ is projective. Now let M be a nonzero torsion free left R-module. Let $(0 \neq) x \in M$. Then $(0 \neq) Rx \subseteq M$, and Rx is cyclic and torsion free. So, Rx is projective. From this it is immediate to see that each torsion free cyclic left R-module is injective if and only if R is left self-injective.

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