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BRIDGE AND CYCLE DEGREES OF VERTICES OF GRAPHS

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ABSTRACT. The bridge degree bdeg v and cycle degree cdeg v of a vertex v in a graph G are, respectively, the number of bridges and number of cycle edges incident with v in G. A characterization of finite nonempty sets S of nonnegative integers is given for which S is the set of bridge degrees (cycle degrees) of the vertices of some graph. The bridge-cycle degree of a vertex v in a graph G is the ordered pair (b,c), where bdeg v = b and cdeg v = c. Those finite sets S of ordered pairs of nonnegative integers for which S is the set of bridge-cycle degrees of the vertices of some graph are also characterized.

KEY WORDS AND PHRASES. Bridge degree, cycle degree, bridge-cycle degree
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1. INTRODUCTION.

Except for the number of vertices and number of edges, probably the most basic numbers associated with a graph are the degrees of its vertices. There are several results on graphs dealing strictly with degrees. In this paper we differentiate between two types of edges, namely bridges and cycle edges, and consider the two associated degrees. We then investigate the related degree sets, both separately and combined. In each case we characterize which sets are the degree sets of connected graphs.

We adopt the graph-theoretic terminology and notation of Behzad, Chartrand, and Lesniak-Foster [1]. For a vertex v of a graph G, we define the bridge degree bdeg v of v as the number of bridges incident with v; the cycle degree cdeg v of v is the number of cycle edges incident with v. Clearly, deg v = bdeg v + cdeg v. If v is neither a cut-vertex nor an end-vertex, then bdeg v = 0. The converse of this is not true, however, since bdeg $v = 0$ for every vertex v of a 2-edge-connected graph.

2. PRELIMINARIES

Similar to the so-called "First Theorem of Graph Theory" (which states that the sum of the degrees of the vertices of a graph is twice the number of edges), we have the following observation.

LEMMA 1. Let G be a graph with b bridges and c cycle edges. Then

$$\sum_{v\in V(G)} \text{bdeg } v = 2b \quad \text{and} \quad \sum_{v\in V(G)} \text{cdeg } v = 2c.$$

PROOF. Since each bridge of G is incident with two vertices of G, each bridge is counted twice in the sum $\sum_{v\in V(G)} \text{bdeg } v$; therefore $\sum_{v\in V(G)} \text{bdeg } v = 2b$. Similarly,

$$\sum_{v\in V(G)} \text{cdeg } v = 2c. \qquad \square$$

The following corollary is an immediate consequence of Lemma 1.

COROLLARY. The number of vertices with odd bridge(cycle) degree is even.

The following lemma is elementary but useful.

LEMMA 2. Let G be a connected graph with at least one bridge. Then there exist at least two vertices with bridge degree 1.

PROOF. Let G' be the subgraph of G induced by the bridges of G. Then G' is a forest and, therefore, every component of G' is a tree (at least one of which is nontrivial) which implies that G' contains at least two vertices of degree 1 in G'. Since the bridge degree of a vertex v in G' is the bridge degree of v in G, it follows that G contains at least two vertices of degree 1. $\square$

While the previous lemma states that in a graph with at least one bridge, at least two vertices have bridge degree 1, it is clear that no vertex has cycle degree 1.

3. BRIDGE AND CYCLE DEGREE SETS.

The *degree set* D_G of a graph G is defined as the set of degrees of the vertices of G. Now similar to the degree set D_G of a graph G, we define the *bridge degree set* B_G of a graph G as the set of bridge degrees of the vertices of G, and the *cycle degree set* C_G of G as the set of cycle degrees of the vertices of G. Clearly B_G and C_G are nonempty sets since bdeg v and cdeg v are defined for every vertex v of G. For a graph G we let $B_G = \{b_1, b_2, \ldots, b_m\}$ and $G_G = \{c_1, c_2, \ldots, c_n\}$, where $m, n \geq 1$ with $b_1 < b_2 < \ldots < b_m$ and $c_1 < c_2 < \ldots < c_n$. As noted earlier, either $B_G = \{0\}$ or $1 \in B_G$, and $1 \notin C_G$.

For a set $S = \{a_1, a_2, \ldots, a_k\}$, $k \geq 1$, of positive integers, where $a_1 < a_2 < \ldots < a_k$, Kapoor, Polimeni and Wall [2] defined $\mu(S) = \mu(a_1, a_2, \ldots, a_k)$ as the minimum order of a graph G for which $D_G = S$. They showed the following.

THEOREM A. For every finite set S of positive integers, $\mu(S)$ exists and, in fact, $\mu(S) = a_k + 1$.

In a similar fashion we introduce two new definitions.

For a finite set S of nonnegative integers, let $\mu_b(S)$ represent the minimum order of graph G for which $B_G = S$. If no such graph G exists, then we write

$\mu_b(S) = +\infty$. If $S = \{b_1, b_2, \ldots, b_m\}$, $m \geq 1$ where $b_1 < b_2 < \ldots < b_m$, we also write $\mu_b(S) = \mu_b(b_1, b_2, \ldots, b_m)$. Since every graph which contains a vertex of bridge degree b_m has order at least $b_m + 1$, it follows that $\mu_b(b_1, b_2, \ldots, b_m) \geq b_m + 1$ whenever $\mu_b(S)$ exists.

Similarly for a finite set S of nonnegative integers, let $\mu_c(S)$ represent the minimum order of graph G for which $C_G = S$. If no such graph G exists, then we write $\mu_c(S) = +\infty$. If $S = \{c_1, c_2, \ldots, c_n\}$, $n \geq 1$, where $c_1 < c_2 < \ldots < c_n$, we write $\mu_c(S) = \mu_c(c_1, c_2, \ldots, c_n)$. Since every graph G with $C_G = S$ contains a vertex of cycle degree c_n, it follows that G has order at least $c_n + 1$, so that $\mu_c(c_1, c_2, \ldots, c_n) \geq c_n + 1$ whenever $\mu_c(S)$ exists.

The following result of Kapoor, Polimeni and Wall [2] will prove to be useful.

THEOREM B. Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, be a set of positive integers.

(i) There exists a tree T with $D_T = S$ if and only if $1 \in S$.

(ii) If $1 \in S$, then the minimum order of a tree T with $D_T = S$ is

$$2 + \sum_{i=1}^{n} (a_i - 1).$$

For forests there is an analogous result which we now present.

PROPOSITION 1. Let $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, be a set of positive integers. There exists a forest F with $D_F = S$ if and only if $1 \in S$. Moreover, if $1 \in S$, then the minimum order of a forest F with $D_F = S$ is

$$2 + \sum_{i=1}^{n} (a_i - 1)$$

and, in fact, F is a tree.

PROOF. By Theorem B(i) there exists a forest F (in fact a tree) with $D_F = S$ if and only if $1 \in S$. Furthermore by Theorem B(ii) the minimum order of a tree with $D_T = S$ is

$$2 + \sum_{i=1}^{n} (a_i - 1).$$

Now suppose that F is any forest of order p and size q with $k \geq 1$ (nontrivial) components for which $D_F = S$, and let m_i be the number of vertices of degree $a_i (1 \leq i \leq n)$ in F. Then

$$2(p - k) = 2q = \sum_{i=1}^{n} m_i a_i \geq (p - n) \cdot 1 + \sum_{i=1}^{n} a_i,$$

which implies that

$$p \geq 2k + \sum_{i=1}^{n} (a_i - 1) \geq 2 + \sum_{i=1}^{n} (a_i - 1)$$

since $k \geq 1$. Therefore the minimum order of a forest F for which $D_F = S$ is

$$2 + \sum_{i=1}^{n} (a_i - 1),$$

which in turn implies that $k = 1$ and that F is a tree. []

As an immediate consequence of Proposition 1 we show that $\mu_b(S)$ exists for every finite nonempty set S of positive integers.

PROPOSITION 2. For each set $S = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, of positive integers with $1 = a_1 < a_2 < \ldots < a_n$, there exists a graph G for which $B_G = S$. Moreover, any such smallest graph G is a tree and has order

$$2 + \sum_{i=1}^{n} (a_i - 1).$$

PROOF. By Proposition 1, there exists a tree T with $D_T = B_T = S$ and have order

$$|V(T)| = 2 + \sum_{i=1}^{n} (a_i - 1),$$

which verifies the first statement of the proposition. Suppose next that G is a graph of order p with $B_G = S$. We show that

$$p \geq 2 + \sum_{i=1}^{n} (a_i - 1).$$

If G contains no cycles, this inequality follows by Proposition 1.

Suppose then that G contains cycles. We remove all cycle edges of G, producing a forest F of order p having the property that the bridge degree of each vertex in both G and F is the same, implying that $B_F = S$. We next show that F is not a tree. Let $e = uv$ be a cycle edge of G. We claim that u and v are not connected in F, for suppose P is a $u - v$ path of F. Then $P + e$ is a cycle in G, implying that no edges of P are present in F, a contradiction. Consequently, F is not a tree and so by Proposition 1,

$$p > 2 + \sum_{i=1}^{n} (a_i - 1). \qquad \square$$

The case where S contains the integer 0 is now treated.

PROPOSITION 3. Let $S = \{b_0, b_1, \ldots, b_n\}$, $n \geq 1$, be a set of nonnegative integers with $b_0 = 0$ and $1 = b_1 < b_2 < \ldots < b_n$. Then there exists a connected graph G such that $B_G = S$. Moreover, the minimum order of such a graph G is

$$4 + \sum_{i=1}^{n} (b_i - 1).$$

PROOF. Consider a tree T with degree set $\{b_1, b_2, \ldots, b_n\}$ and of order

$$2 + \sum_{i=1}^{n} (b_i - 1).$$

Then construct a graph G by identifying a vertex of degree 1 in T and a vertex of K_3. Clearly,

$$|V(G)| = 4 + \sum_{i=1}^{n} (b_i - 1)$$

and $B_G = S$. Suppose G_1 is a connected graph of minimum order having $B_G = S$. Let F be the forest obtained from G_1 by removing the cycle edges of G_1, and F_1 the forest obtained from F by removing the vertices of F having degree 0. Necessarily the degree of each vertex of F_1 is the same as the bridge degree of that vertex in F

and G_1. Then

$$|V(G_1)| = |V(F)| > |V(F_1)| \geq 2 + \sum_{i=1}^{n} (b_i - 1).$$

Hence

$$|V(G_1)| \geq 3 + \sum_{i=1}^{n} (b_i - 1).$$

If equality holds here, then

$$|V(F_1)| = 2 + \sum_{i=1}^{n} (b_i - 1)$$

which implies that F_1 is a tree (by Proposition 2) and G_1 contains one vertex, say v, whth $\text{bdeg } v = 0$. Let $e_1 = vw_1$ and $e_2 = vw_2$ be two edges incident with v. Observe that every edge of the $w_1 - w_2$ path P in F_1 is a bridge while every edge of P in $F + e_1 + e_2$ and therefore in G is a cycle edge, but this is impossible. Therefore

$$|V(G_1)| = 4 + \sum_{i=1}^{n} (b_i - 1).$$

We now consider the existence of $\mu_c(S)$ for finite nonempty sets S of nonnegative integers.

PROPOSITION 4. For each set $S = \{c_1, c_2, \ldots, c_n\}$, $n \geq 1$, of positive integers with $2 \leq c_1 < c_2 < \ldots < c_n$, there exists a connected graph G (having no bridges) for which $C_G = S$. Moreover, any such smallest graph G has order $c_n + 1$.

PROOF. We follow the techique employed in [2] used in proving Theorem A. Clearly a graph G with $C_G = S$ has at least $c_n + 1$ vertices. We show there exists a graph G with $c_n + 1$ vertices having $C_G = S$. If $n = 1$, then $G = K_{c_1 + 1}$ has the desired properties; while if $n = 2$, then $G = K_{c_1} + \bar{K}_{c_2 - c_1 + 1}$ has order $c_2 + 1$ and $C_G = S$.

Suppose $n > 3$. By Theorem A there exists a graph H of order $c_{n-1} - c_1 + 1$ such that $D_H = \{c_2 - c_1, c_3 - c_1, \ldots, c_{n-1} - c_1\}$. The graph $G = K_{c_1} + (\bar{K}_{c_n - c_{n-1}} \cup H)$ is connected (in fact, 2-edge-connected), has order $c_n + 1$ and $C_G = D_G = \{c_1, c_2, \ldots, c_n\}$. □

The following Proposition is now immediate.

PROPOSITION 5. For each set $S = \{c_0, c_1, \ldots, c_n\}$, $n \geq 1$, of nonnegative integers with $c_0 = 0$ and $2 \leq c_1 < \ldots < c_n$, there exists a connected graph G having $C_G = S$. Moreover, any such smallest graph G has order $c_n + 2$ which implies that $\mu_c(S) = c_n + 2$.

We now determine conditions on two given sets B and C such that there exists a connected graph G with $B_C = B$ and $C_G = C$.

THEOREM 1. Given two finite nonempty sets B and C of nonnegative integers, there exists a connected graph G having $B_G = B$ and $C_G = C$ if and only if exactly one of the following statements holds:

(1) $C = \{0\}$, $1 \in B$ and $0 \notin B$,

(2) $C = B = \{0\}$,

(3) $1 \in B - C$, $0 \in C$ and $C \neq \{0\}$,

(4) $0, 1 \notin C$ and $B = \{0\}$,

(5) $0, 1 \in B - C$.

PROOF. If (1) holds, then by Proposition 2, there exists a tree T with $B_T = B$ (and, of course, $C_T = \{0\}$). If (2) holds, then K_1 has the desired properties. If (3) holds, we consider the following cases.

CASE 1. Suppose $0 \in B$. Then by Proposition 4 there exists a bridgeless connected graph H with $C_H = C - \{0\}$ and by Proposition 2 there exists a tree T with $B_T = B - \{0\}$. Now we construct a graph G having $B_G = B$ and $C_G = C$ by identifying a vertex of H and an end-vertex of T.

CASE 2. Suppose $0 \notin B$. We consider the same graph G produced in Case 1 (noting that $B - \{0\} = B$). Then we construct a graph H by joining each vertex v_α of G having bridge degree 0 to a new vertex w_α. Observe that $B_H = B$ and $C_H = C$.

If (4) holds then the graph G in Proposition 4 has the property that $B_G = B$ and $C_G = C$. If (5) holds then by Proposition 2 there exists a tree T having $B_T = B - \{0\}$ and by Proposition 4 there exists a graph H having $C_H = C$. Now we construct a graph G having $B_G = B$ and $C_G = C$ by identifying each vertex v_α of T with a vertex of a copy H_α of H.

Conversely, we show that given two sets of nonnegative integers B and C and a connected graph G having $B_G = B$ and $C_G = C$, then exactly one of the statements (1) through (5) must hold. Suppose, to the contrary, that none of the statements (1) through (5) holds. Clearly $1 \notin C_G$. Now if $0 \in C$, then either (a) $C = \{0\}$ and $0, 1 \in B$ (implying that G has a vertex of degree 0 and a bridge, which contradicts the fact that G is connected) or (b) $C \neq \{0\}$ and $B = \{0\}$ (a similar contradiction). If $0 \notin B, C$ and $1 \in B - C$, then we consider a longest path P in G. Let u be an end-vertex of P and e be the edge incident with u in P.

Suppose that e is a bridge of G. Then there must exist a cycle edge uw in G for some w in $V(G)$ since $0 \notin C$. Now since e is a beidge of G then w cannot be a vertex of P, which implies that there exists a path longer than P which is a contradiction. If e is a cycle edge, then, in a similar fashion, we reach a contradiction.

Therefore there does not exist a graph G having $B_G = B$ and $C_G = C$ for which $0 \notin B, C$ and $1 \in B - C$. □

4. BRIDGE-CYCLE DEGREE SETS.

We have determined in Section 3 which bridge degrees and which cycle degrees are possible for the vertices of some connected graph. We now consider the problem of specifying a given bridge degree <u>and</u> cycle degree for some vertex of a connected graph.

If v is a vertex of a graph G having $\text{bdeg } v = b$ and $\text{cdeg } v = c$, then the

<u>bridge-cycle degree</u> bcdeg v of v is the ordered pair (b,c). The <u>bridge-cycle degree set</u> BC_G of G is the set of bridge-cycle degrees of the vertices of G. In this section we determine those finite sets of ordered pairs of nonnegative integers that are the bridge-cycle degree sets of some graph. For the purpose of doing this, we begin with three lemmas that specify the existence of certain kinds of graphs having prescribed degree conditions.

LEMMA 3. For each integer $m \geq 2$ and nonnegative integers $n_2, n_3, \ldots, n_m$, there exists a tree having exactly n_i vertices of degree i for $i = 2,3,\ldots,m$.

PROOF. Such a tree can be constructed by taking a path P of order

$$\sum_{i=2}^{m} n_i$$

and attaching sufficiently many terminal bridges (and their corresponding end-vertices) to the vertices of P to produce the desired degrees.

Let u_1v_1 be an edge of a graph G_1 and u_2v_2 an edge of a graph G_2 disjoint from G_1. By a <u>2-edge transfer of</u> $G_1 \cup G_2$ <u>at</u> u_1v_1 <u>and</u> u_2v_2, we mean the graph $G_1 \cup G_2 - u_1v_1 - u_2v_2 + u_1u_2 + v_1v_2$ or the graph $G_1 \cup G_2 - u_1v_1 - u_2v_2 + u_1v_2 + u_2v_1$. We note that if G_1 and G_2 are 2-edge-connected, then so too is the 2-edge transfer of $G_1 \cup G_2$ at any edge of G_1 and any edge of G_2.

LEMMA 4. Let D be a finite set of integers each element of which is at least 2, and for each $i \in D$, let n_i be a positive integer. Then there exists a 2-edge connected graph with degree set D and having at least n_i vertices of degree i for each $i \in D$.

PROOF. Let r be the smallest element of D, and let k be the smallest positive integer such that $k(r+1) \geq n_r$. Define $G_1 = G_2 = \ldots = G_k = K_{r+1}$. Next let s be the smallest integer $i \in D$ such that $i > r$ (if such an integer exists). Let ℓ be the smallest positive integer such that $\ell(s+1) \geq n_s$ and define $G_{k+1} = G_{k+2} = \ldots + G_{k+\ell} = K_{s+1}$. Continuing in this fashion, we arrive at a sequence $G_1, G_2, \ldots, G_z$ of complete graphs. For $i = 1,2,\ldots,z-1$, we construct a 2-edge transfer of $G_i \cup G_{i+1}$, selecting disjoint edges of G_i for $i = 2,3,\ldots,z-1$ in the process. The resulting graph has the desired properties. □

LEMMA 5. Let $m \geq 2$ and $n \geq 2$ be (not necessarily distinct) integers where n is odd if m is odd. There exists a 2-edge-connected graph with one vertex of degree m and the remaining vertices of degree n.

PROOF. If $m = n$, then K_{n+1} has the desired properties; thus assume $m \neq n$. Let $G_1, G_2, \ldots, G_z$ be pairwise disjoint graphs isomorphic to K_{n+1}, where z is sufficiently large so that $G_1 \cup G_2 \cup \ldots \cup G_z$ has a set W of $m/2$ independent edges if m is even or $(m+n-2)/2$ independent edges if m and n are odd. For $i = 1,2,\ldots,z-1$, we can construct a 2-edge transfer of $G_i \cup G_{i+1}$ at edges of G_i and G_{i+1} not in W, producing a graph G. Delete the edges of W from G. If m is even, we add a new vertex v to $G - W$ and join v to those m vertices of $G - W$ having degree $n - 1$. If m and n are odd, we add two adjacent vertices u and w to $G - W$, joining u, say, to $m - 1$ vertices of degree $n - 1$ and joining w to $n - 1$

vertices of degree $n - 1$. In either case, the graph produced has the required properties. □

Next we present three additional lemmas that will be needed in the main theorem to follow.

LEMMA 6. Let G be a connected graph such that $(a,0),(0,b) \in BC_G$, where $a \geq 1$ and $b \geq 2$. Then there exist integers $c \geq 1$ and $d \geq 2$ such that $(c,d) \in BC_G$.

PROOF. Let u and v be vertices of G having bridge-cycle degrees $(a,0)$ and $(0,b)$ respectively. Let P be a $u - v$ path in G. Let w be the last vertex of G on P having cycle degree 0 and let x be the next vertex on P. Since w is incident only with bridges, xw is a bridge. Since x has positive cycle degree, $\text{bcdeg}\ x = (c,d)$ for some $c \geq 1$ and $d \geq 2$. □

LEMMA 7. Let G be a nontrivial connected graph such that $B_G \neq \{0\}$ and $(1,0) \notin BC_G$. Then (i) $(1,n) \in BC_G$ for some even positive integer n or $(0,n) \in BC_G$ for some odd integer $n \geq 3$, and (ii) $(0,n) \in BC_G$ for some integer $n \geq 2$.

PROOF. Contract each cyclic (2-edge-connected) block of G to a vertex producing a graph H. Necessarily, H is a tree. Since $B_G \neq \{0\}$, there exists an end-vertex v in H. Let e be the edge of H incident with v. We may either consider v to be (a) a vertex of G or (b) the vertex obtained by contracting a cyclic block J of G, with a vertex u of J incident with e. If v is a vertex of G, then $\text{bcdeg}\ v = (1,0)$ which produces a contradiction. Hence the situation (b) must occur. If $\deg_J u$ is odd, then the degree of at least one other vertex of J is odd and so $(0,n) \in BC_G$ for some odd integer $n \geq 3$; otherwise, $\text{bcdeg}\ u = (1,n)$ for some even positive integer n. Hence (i) is established. In either case, the bridge-cycle degree of any vertex in J different from u is $(0,n)$ for some integer $n \geq 2$, verifying (ii). □

Our next lemma actually reiterates some earlier observations; nevertheless, it is useful to include it.

LEMMA 8. Let G be a nontrivial connected graph. Then (i) $(0,0) \notin BC_G$, (ii) $1 \notin C_G$, and (iii) if $n \in B_G$ for some positive integer n, then $1 \in B$.

PROOF. Since G is connected and nontrivial, (i) follows immediately. We have noted the necessity of (ii) and (iii) earlier, the latter following from Lemma 2. □

We are now prepared to present the main result of this section.

THEOREM 2. Let S be a finite nonempty set of ordered pairs of nonnegative integers and let $B = \{b \mid (b,c) \in S$ for some $c\}$ and $C = \{c \mid (b,c) \in S$ for some $b\}$. Then there exists a nontrivial connected graph G with $BC_G = S$ if and only if the following conditions are satisfied:

(a) if $(a,0)$, $(0,b) \in S$, where $a \geq 1$ and $b \geq 2$, then there exist integers $c \geq 1$ and $d \geq 2$ such that $(c,d) \in S$;

(b) if $B \neq \{0\}$ and $(1,0) \notin S$, then

(i) $(1,n) \in S$ for some even positive integer n or $(0,n) \in S$ for some odd integer $n \geq 3$, and

(ii) $(0,n) \in S$ for some integer $n \geq 2$;

(c)(iii) $(0,0) \notin S$, (iv) $1 \notin C$, and (v) if $n \in B$ for some positive integer n, then $1 \in B$.

PROOF. The necessity of the conditions (a), (b) and (c) follow directly from Lemmas 6, 7 and 8.

To verify the sufficiency, let S be a finite nonempty set of ordered pairs of nonnegative integers satisfying (a), (b) and (c). We show the existence of a nontrivial connected graph G for which $BC_G = S$.

Note by (c) that each element of S is of the form $(m,0), m \geq 1$, or $(0,n)$, $n \geq 2$, or (m,n), where $m \geq 1$ and $n \geq 2$. We now consider two cases, depending on whether $(1,0)$ does or does not belong to S.

CASE 1. Assume $(1,0) \in S$. By Lemma 3, there exists a tree T_1 with exactly one vertex of degree i for each $i \geq 2$ such that $(i,0) \in S$. If there is no such i, then let $T_1 = K_2$. If $C = \{0\}$, then $BC_{T_1} = S$ so that T_1 is the desired graph.

Assume then that $C \neq \{0\}$ and let $D = C - \{0\}$. Thus $D \neq \phi$ and each integer of D is at least 2. For $i \in D$, let

$$n_i = |\{j \mid (j,i) \in S\}|.$$

By Lemma 4, there exists a 2-edge-connected graph F with degree set D and having at least n_i vertices of degree i for each $i \in D$. For $(j,i) \in S$, let $u_{j,i}$ be a vertex of F such that $\deg_F u_{j,i} = i$. By (a), there exist integers $c \geq 1$ and $d \geq 2$ such that $(c,d) \in S$. Identify an end-vertex of T_1 with $u_{c,d}$ in F. Now attach sufficiently many terminal bridges to the vertices of F, if required, to obtain a graph G in which the bridge-cycle degree of each vertex belongs to S and $\operatorname{bcdeg}_G u_{j,i} = (j,i)$. Then $BC_G = S$.

CASE 2. Assume $(1,0) \notin S$. If $B = \{0\}$, then the existence of a connected graph G with $BC_G = S$ is guaranteed by Lemma 4.

Suppose then that $B \neq \{0\}$. By Lemma 3, there exists a tree T_2 with at least one vertex of degree j (each labeled $v_{j,i}$) for each $(j,i) \in S$ such that $j \geq 1$.

Next we proceed to construct a graph H with the desired bridge-cycle degrees, with the possible exception of some elements of S of the form $(0,n)$ $n \geq 2$.

CASE 2a. Assume $(0,n_1) \in S$ for some odd integer $n_1 \geq 3$. Since $B \neq \{0\}$ and $(1,0) \notin S$, it follows from condition (v) that $(1,n_2) \in S$ for some integer $n_2 \geq 2$. For each $(j,i) \in S$, where $j \geq 1$ and $i \geq 2$, there exists by Lemma 5, a 2-edge-connected graph $F_{j,i}$ having one vertex $u_{j,i}$ of degree i and the remainder of degree n_1. The graph H is now produced by identifying, for each $(j,i) \in S$, $j \geq 1$, $i \geq 2$, every vertex $V_{j,i}$ of T_2 with the vertex $u_{j,i}$ of a copy of $F_{j,i}$.

CASE 2b. Assume there exists no odd integer $n \geq 3$ such that $(0,n) \in S$. Then by condition (ii) $(0, n_1) \in S$ for some integer $n_1 \geq 2$; and by (i), $(1, n_2) \in S$ for some even integer $n_2 \geq 2$.

Consider $(j,i) \in S$, where $j \geq 1$ and $i \geq 2$. If i is even, we obtain the graph $F_{j,i}$ as in Case 2a, and identify each vertex $v_{j,i}$ of T_2 with $u_{j,i}$ in $F_{j,i}$. If i is odd, join $v_{j,i}$ to each vertex in a copy of K_i. Now we construct ij mutually disjoint copies of a 2-edge-connected graph $F'_{j,i}$ with one vertex of degree n_2 and the remainder of degree n_1 (see Lemma 5). We next join each vertex of the graph K_i considered above with the vertex of degree n_2 in j of the copies $F'_{j,i}$, using each copy of $F'_{j,i}$ for just one vertex of K_i. We then repeat this process, using separate copies of the graph K_i, for each $(j,i) \in S$, $j \geq 1$, $i \geq 2$, to obtain the desired graph H.

As mentioned above, the graph H, constructed in either Case 2a or 2b, has the appropriate bridge-cycle degrees, with the possible exception of some elements of S of the form $(0,n)$, $n \geq 2$. In particular, if $(0,n) \in S$ for some $n \neq n_1$, then we employ Lemma 4 to obrain a 2-edge-connected graph H' whose degree set consists of all such integers n. A 2-edge transfer of $H \cup H'$ at an edge of H' and a cycle edge of H is a connected graph G such that $BC_G = S$. □

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