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RESEARCH NOTES

PERIODIC RINGS WITH COMMUTING NILPOTENTS

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ABSTRACT. Let R be a ring (not necessarily with identity) and let N denote the set of nilpotent elements of R. Suppose that (i) N is commutative, (ii) for every x in R, there exists a positive integer $k = k(x)$ and a polynomial $f(\lambda) = f_x(\lambda)$ with integer coefficients such that $x^k = x^{k+1}f(x)$, (iii) the set $I_n = \{x \mid x^n = x\}$ where n is a fixed integer, $n > 1$, is an ideal in R. Then R is a subdirect sum of finite fields of at most n elements and a nil commutative ring. This theorem generalizes the "$x^n = x$" theorem of Jacobson, and (taking $n = 2$) also yields the well known structure of a Boolean ring. An Example is given which shows that this theorem need not be true if we merely assume that I_n is a subring of R.

KEY WORDS AND PHRASES. Boolean ring, subdirect sum, subdirectly irreducible.
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1. INTRODUCTION.

A well known theorem of Jacobson [1] states that a ring R satisfying the identity $x^n = x$, $n > 1$ is fixed, is a subdirect sum of finite fields of at most n elements. Such rings, of course, have no nonzero nilpotents. With this as motivation, we consider the structure of a "periodic" ring R with commuting nilpotents and for which the set $I_n = \{x \mid x^n = x\}$ forms an ideal in R. We show that such a ring R has a structure similar to that given in Jacobson's Theorem. As a corollary, we show that by taking $n = 2$, we recover the familiar structure of a Boolean ring (as a subdirect sum of copies of GF(2)). Finally, we give an example which shows that this theorem need not be true if we assume that I_n is merely a subring of R (instead of an ideal).

2. MAIN RESULTS.

Our main result is the following

MAIN THEOREM. Let R be a ring (not necessarily with 1), and let N be the set of nilpotents of R. Suppose that (i) N is commutative, (ii) for every x in R, there exists a positive integer $k = k(x)$ and a polynomial $f(\lambda) = f_x(\lambda)$ with integer coefficients such that $x^k = x^{k+1}f(x)$, (iii) the set $I_n = \{x \mid x \in R, x^n = x\}$ where n is a fixed integer, $n > 1$, is an ideal in R. Then R is commutative and, in fact, R is a subdirect sum of fields of at most n elements and a nil commutative ring.

PROOF. The proof will be broken into several claims.

CLAIM 1. The idempotents of R are all in the center of R.

For, suppose $e^2 = e \in R$, $x \in R$. Then $e \in I_n$ and hence, by (iii), $ex - exe \in I_n$; that is,

$$ex - exe = (ex - exe)^n = 0$$

and hence $ex = exe$. Similarly, $xe = exe$, which proves Claim 1.

CLAIM 2. If $\phi : R \to R^*$ is an onto homomorphism, then $\phi(N)$ coincides with the set of all nilpotent elements of R^*.

This was proved by Abu-Khuzam and Yaqub [2] and by Ikehata and Tominaga [3], but for convenience we reproduce the proof. Let d^* be an arbitrary nilpotent element of R^* with $(d^*)^m = 0$. Let $d \in R$ be such that $\phi(d) = d^*$. By (ii), $d^k = d^{k+1}f(d)$ for some positive integer k (depending on d) and some polynomial $f(\lambda)$ with integer coefficients (again depending on d). The last equation implies that

$$[d - d^2f(d)]^k = 0 \text{ and hence } d - d^2f(d) \in N. \tag{2.1}$$

Observe that

$$d - d^{m+1}(f(d))^m = (d - d^2f(d)) + (df(d))(d - d^2f(d)) + \dots + (df(d))^{m-1}(d - d^2f(d))$$

and hence, by (2.1),

$$d - d^{m+1}(f(d))^m \in N. \tag{2.2}$$

Recalling that $\phi(d) = d^*$ and $(d^*)^m = 0$, (2.2) implies

$d^* = \phi(d - d^{m+1}(f(d))^m) \in \phi(N)$; that is, $d^* \in \phi(N)$. This proves Claim 2.

CLAIM 3. Hypothesis (ii) implies that $(xf(x))^k$ is idempotent and, moreover, $x^k = x^k\{xf(x)\}^k$.

For, $x^k = x^{k+1}f(x)$ implies (by multiplying both sides by $xf(x)$ a suitable number of times) $x^k = x^{k+r}(f(x))^r$ for all positive integers r, and hence, in particular, $x^k = x^{2k}(f(x))^k$. Note that $(xf(x))^k$ is indempotent, and Claim 3 is proved.

To complete the proof of the Main Theorem, first recall that

$$R \cong \text{a subdirect sum of rings } R_i (i \in \Gamma);$$

each R_i is subdirectly irreducible. Let

$$\phi_i : R \to R_i$$

be the natural homomorphism of R onto R_i. We now distinguish two cases.

CASE 1: R_i does not have an identity. Let $x_i \in R_i$ and let $\phi(x) = x_i$, $x \in R$. By Claims 3 and 1, $(xf(x))^k$ is a central idempotent in R and hence $(x_i f(x_i))^k$ is a central idempotent in the subdirectly irreducible ring R_i: Therefore, $(x_i f(x_i))^k = 0$ and hence by Claim 3, $x_i^k = 0$. Thus, R_i is a nil ring. Moreover, by Claim 2,

$$\phi_i(N) = \text{nilpotents of } R_i = R_i \text{ (since } R_i \text{ is nil).} \tag{2.3}$$

But, by (i), N is commutative and hence by (2.3), $\phi_i(N)[= R_i]$ is commutative. In other words, R_i is a nil commutative ring in this case.

CASE 2: R_i has an identity 1.

As we saw in Case 1, for any x_i in R_i, $(x_i f(x_i))^k$ is a central idempotent and hence (since R_i is subdirectly irreducible)

$$(x_i f(x_i))^k = 0 \quad \text{or} \quad (x_i f(x_i))^k = 1. \tag{2.4}$$

If for some x_i in R_i, $(x_i f(x_i))^k = 0$ then by Claim 3, $x_i^k = 0$ and thus x_i is nilpotent. On the other hand, if $(x_i f(x_i))^k = 1$ then x_i is a unit. We have thus shown that

$$x_i \text{ is nilpotent or } x_i \text{ is a unit, for all } x_i \text{ in } R_i. \tag{2.5}$$

Let $I_i^* = \phi_i(I_n)$. Then I_i^* is an ideal in R_i. Let $x_i \in I_i^*$ and thus $x_i = \phi_i(x)$ for some $x \in R$ with $x^n = x$. Therefore,

$$x_i^n = x_i \quad \text{for all} \quad x_i \in I_i^*. \tag{2.6}$$

Let $e \in R$ be such that $\phi_i(e) = 1$. By (ii), $e^k = e^{k+1}f(e)$ and hence $\phi_i(e^k) = \phi_i(e^{k+1}f(e))$. Thus, $1 = 1 \cdot \phi_i(f(e)) = f(\phi_i(e)) = f(1)$. Moreover, since $(ef(e))^k$ is idempotent (Claim 3), $(ef(e))^k \in I_n$ and hence

$$(\phi_i(e)\, \phi_i\, (f(e)))^k \in I_i^* \; [= \phi_i(I_n)]. \tag{2.7}$$

Now, since $\phi_i(e) = 1$ and $\phi_i(f(e)) = f(\phi_i(e)) = f(1) = 1$ (as shown above), (2.7) implies that $1 \in I_i^*$ and hence $R_i = I_i^*$ (since I_i^* is an ideal). Combining this with (2.6), we see that

$$x_i^n = x_i \quad \text{for all} \quad x_i \in R_i. \tag{2.8}$$

Combinbing (2.5) and (2.8), we conclude that R_i is a division ring satisfying the identity in (2.8), and hence by Jacobson's Theorem [1], R_i is a field with at most n elements (since n is fixed). This completes the proof of the Main Theorem.

Taking $n = 2$ in our Main Theorem, we get

COROLLARY 1. *Let* R *be a ring and* N *the set of nilpotents of* R. *Suppose that* (i) N *is commutative*, (ii) *for every* x *in* R, *there exists a positive integer* k *and a polynomial* $f(\lambda)$ *with integer coefficients such that* $x^k = x^{k+1}f(x)$, (iii) *the idempotents of* R *form an ideal in* R. *Then* R *is com-*

mutative and, in fact, R is a subdirect sum of copies of GF(2) and a nil commutative ring.

As a further corollary of our Main Theorem, we obtain Jacobson's Theorem [1]:

COROLLARY 2. *Let* R *be a ring satisfying the identity* $x^n = x$, *where* $n > 1$ *is a fixed integer. Then* R *is a subdirect sum of finite fields each of which has at most* n *elements*.

Taking $n = 2$ in Corollary 2, we also obtain the following

COROLLARY 3. *A Boolean ring is a subdirect sum of copies of* GF(2).

We conclude with the following example which shows that our Main Theorem need not be true if we merely assume that I_n is a *subring* of R.

EXAMPLE. Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \; a, b, c \in GF(4) \right\}.$$

Note that the set E of all idempotents is just $\{0, 1\}$ and thus E is a *subring* of R (since R is of characteristic 2). It is readily verified that $N^2 = \{0\}$ and $x^8 = x^2$ for all x in R, and hence all the hypotheses of our Main Theorem are satisfied *except* that the subring E is *not* an ideal. Observe that R is *not* commutative.

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