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ON AN INTEGRAL INEQUALITY IN N-INDEPENDENT VARIABLES

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ABSTRACT. We present a new non-linear integral inequality of the Gronwall-Bellman-Bihari type in n-independent variables with application to pointwise estimates of solutions of a certain class of non-linear hyperbolic partial differential equation.

KEY WORDS AND PHRASES. *Integral inequality in n-variables, Gronwall-Bellman-Bihari type inequality, non-linear hyperbolic partial differential equations.*

AMS (MOS) SUBJECT CLASSIFICATIONS (1980). *Primary 26D10; Secondary 35B35, 35B45.*

1. INTRODUCTION.

There have been various linear and non-linear generalizations of the celebrated Gronwall-Bellman inequality [see 1-10]. These generalizations have been largely motivated by specific applications of the inequality to ordinary differential and integral equations in proving uniqueness, boundedness, comparison, continuous dependence, perturbations, and stability results. The two independent variable generalization of this inequality due to Wendroff [1, p. 154] has generated a considerable amount of interest, judging by the papers of Bondge and Pachpatte [11, 12, 13], Bondge et all [14], Pachpatte [15-18], Rasmussen [19], Snow [20] and many others. Most recently, the n-independent variable generalization of the Gronwall-Bellman inequality has attracted the interest of Mathematicians. Chandra and Davis [4], Conlan and Diaz [21], Headley [22], Pachpatte [17, 23], Singare and Pachpatte [24], Shih and Yeh [25], Yeh [26], Young [27], Zahariev and Bainov [28], and many others have established several versions of integral inequalities in n-independent variables and exhibited their usefulness in the analysis of various problems in the theory of partial differential and integral equations. In this paper, we present a new non-linear integral inequality of the Bihari type in n-independent variables which generalizes a recent result in n-independent variables of an inequality due to Pachpatte [23]. The inequality of Pachpatte contains only two non-linear terms on the right hand side, while our result is obtained for any finite number of non-linear terms. We illustrate the usefulness of our result.

2. MAIN RESULT.

Let Ω be an open bounded set in the n-dimensional Euclidean space R^n. For arbitrary points $x,y \in R^n$, with $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots y_n)$, we write $x < y$ (or $x \le y$) if and only if $x_i < y_i$ (or $x_i \le y_i$) for $1 \le i \le n$. Let x^0 and x be two arbitrary points in Ω such that $x^0 < x$ and denote by $\int_{x^0}^{x}\dots d\eta$ the n-fold integral $\int_{x_1^0}^{x_1}\dots\int_{x_n^0}^{x_n}\dots d\eta_1 d\eta_2\dots d\eta_n$.

Define $D_i = \frac{\partial}{\partial x_i}$ for $i = 1,2,\dots,n$ and assume that the following hypotheses hold:

(H_1) $f:\Omega \to R = (-\infty,\infty)$ is a positive function, continuous and non-decreasing in $x \in \Omega$.

(H_2) $\phi:\Omega \to R$, $g_j:\Omega \to R$ $j = 1,2,\dots,m$ are functions which are non-negative and continuous on Ω.

(H_3) The function $q:\Omega \to R$ is continuous on Ω and $q(x) \ge 1$ for all x.

(H_4) The function $W:\Omega \times R \to R$ is continuous, non-negative, and non-decreasing in the last variable.

(H_5) The function $K:\Omega \times \Omega \times R \to R$ is continuous, non-negative and non-decreasing in the last variable. In addition, it is uniformly Lipschitz in the last variable.

(H_6) The functions $H_j:R^+ \to R^+ = [0,\infty)$, $j = 1,2,\dots,m$ are all positive, non-decreasing, continuous, and satisfy

(i) $\frac{1}{\alpha} H_j(v) \le H_j\left(\frac{v}{\alpha}\right)$, $v > 0$, $\alpha \ge 1$.

(ii) $H_j(v)$ is submultiplicative for $v \ge 0$ $j = 1,2,\dots,m$.

REMARK: If $H(v) = v^\beta$ with $0 < \beta < 1$ or $H(v) = \sum_{\ell=1}^{p} v^{\beta_\ell}$, $0 < \beta_\ell < 1$, then H satisfies hypothesis (H_6).

In a very recent paper Pachpatte [23] established the following integral inequality among other results.

THEOREM A. Suppose (H_1) and (H_3) - (H_5) are true and let g_j and ϕ be as defined in (H_2) with $j = 1$. If (H_6) is true for $j = 1$ and

$$\phi(x) \le f(x) + q(x)\left(\int_{x^0}^{x} g_1(y)H_1(\phi(y))dy\right) + W\left(x,\int_{x^0}^{x} K(x,y,\phi(y))dy\right) \quad (2.1)$$

is satisfied for all $x \in \Omega$, then, for $x \in \Omega_1 \subset \Omega$,

$$\phi(x) \leq E_1(x)\left\{f(x) + W(x,r(x))\right\} , \tag{2.2}$$

where

$$E_1(x) = q(x)G^{-1}\left[G(1) + \int_{x^0}^{x} g_1(y)H_1(q(y))dy\right] \tag{2.3}$$

in which

$$G(v) = \int_{v^0}^{v} \frac{ds}{H_1(s)} \qquad v > v^0 > 0; \tag{2.4}$$

G^{-1} is the inverse of G such that $G(1) + \int_{x^0}^{x} g_1(y)H_1(q(y))dy \in \mathrm{Dom}(G^{-1})$, for all $x \in \Omega_1$ and $r(x)$, is a solution of the equation

$$r(x) = \int_{x^0}^{x} K(x,y,E_1(y)\left\{f(y) + W(y,r(y))\right\})dy \tag{2.5}$$

existing on Ω.

We now establish an interesting and useful n-independent variable generalization of Theorem A. We observe that while Pachpatte's result contains two non-linear terms in (2.1) we shall present a result which extends the non-linear terms to any finite number.

THEOREM 1. Let (H_1) and (H_3) - (H_5) hold and suppose ϕ and g_j, $j = 1,2,\ldots,m$, are as defined in (H_2). Assume that H_j, $j = 1,2,\ldots,m$, satisfy (H_6); if

$$\phi(x) \leq f(x) + q(x)\sum_{\ell=1}^{m}\left(\int_{x^0}^{x} g_\ell(y)H_\ell(\phi(y))dy\right) + W(x,\int_{x^0}^{x} K(x,y,\phi(y))dy) \tag{2.6}$$

is satisfied for all $x \in \Omega$, then, for $x \in \Omega_1 \subset \Omega$,

$$\phi(x) \leq \left\{f(x) + W(x,R(x))\right\}\prod_{\ell=1}^{m} E_\ell(x), \tag{2.7}$$

where the functions G_ℓ are defined as

$$G_\ell(u) = \int_{u^0}^{u} \frac{ds}{H_\ell(s)} , \quad 0 < u^0 < u, \quad \ell = 1,2,\ldots,m \tag{2.8}$$

with

$$E_1(x) = q(x)G_1^{-1}\left[G_1(1) + \int_{x^0}^{x} g_1(s)H_1(q(s))ds\right] \tag{2.9}$$

and

$$E_\ell(x) = q(x)G_\ell^{-1}\left[G_\ell(1) + \int_{x^0}^{x} g_\ell(s)\prod_{i=1}^{\ell-1} E_i(s)H_\ell(q(s))ds\right], \ \ell = 2,\ldots,m; \tag{2.10}$$

G_ℓ^{-1} is the inverse of G_ℓ such that

$G_\ell(1) + \int_{x_0}^{x} g_\ell(s) \prod_{i=1}^{\ell-1} E_i(s) H_\ell(q(s))ds \in \mathrm{Dom}(G_\ell^{-1})$ and $R(x)$ is a solution of the integral equation

$$V(x) = \int_{x_0}^{x} K(x,y, \prod_{\ell=1}^{m} E_\ell(y)\{f(y) + W(y,V(y))\})dy. \tag{2.11}$$

PROOF. If $m = 1$, then (2.6) becomes (2.1) and Theorem A implies that inequality (2.7) is true if (2.6) holds. We now proceed by induction and assume that inequality (2.6) implies (2.7) is true for k where $1 \le k \le m-1$. Then this means

$$\phi(x) \le f(x) + q(x) \sum_{\ell=1}^{k} \left(\int_{x_0}^{x} g_\ell(y) H_\ell(\phi(y))dy \right)$$

$$+ W(x, \int_{x_0}^{x} K(x,y,\phi(y))dy) \tag{2.12}$$

implies

$$\phi(x) \le \prod_{\ell=1}^{k} E_\ell(x)\{f(x) + W(x,R(x))\} \tag{2.13}$$

where $E_1(x) = q(x)G_1^{-1}\left[G_1(1) + \int_{x_0}^{x} g_1(y)H_1(q(y))dy\right]$,

and

$$E_\ell(x) = q(x)G_\ell^{-1}\left[G_\ell(1) + \int_{x_0}^{x} g_\ell(y) \prod_{j=1}^{\ell-1} E_j(y)H_\ell(q(y))dy\right]$$

for $\ell = 2,3,\dots,k$. G_ℓ^{-1} is the inverse of G_ℓ such that

$G_\ell(1) + \int_{x_0}^{x} g_\ell(y) \prod_{j=1}^{\ell-1} E_j(y)H_\ell(q(y))dy \in \mathrm{Dom}(G_\ell^{-1})$ for $\ell = 1,2,\dots,k$, and $R(x)$ is a solution of the integral equation

$$R(x) = \int_{x_0}^{x} K(x,y, \prod_{\ell=1}^{k} E_\ell(y)\{f(y) + W(y,R(y))\})dy. \tag{2.14}$$

Now assume that (2.6) holds for $m = k+1$; then

$$\phi(x) \le f(x) + q(x) \sum_{\ell=1}^{k+1} \left(\int_{x_0}^{x} g_\ell(y) H_\ell(\phi(y))dy \right) + W(x, \int_{x_0}^{x} K(x,y,\phi(y))dy)$$

$$\le f(x) + q(x) \sum_{\ell=1}^{k} \left(\int_{x_0}^{x} g_\ell(y) H_\ell(\phi(y))dy \right) + q(x) \int_{x_0}^{x} g_{k+1}(y) H_{k+1}(\phi(y))dy$$

$$+ W(x, \int_{x_0}^{x} K(x,y,\phi(y))dy) . \tag{2.15}$$

Define

$$u(x) = f(x) + q(x) \int_{x_0}^{x} g_{k+1}(y) H_{k+1}(\phi(y))dy.$$

Then (2.15) becomes

$$\phi(x) \le u(x) + q(x) \sum_{\ell=1}^{k} \left(\int_{x^0}^{x} g_\ell(y) H_\ell(\phi(y))dy \right) + W(x, \int_{x^0}^{x} K(x,y,\phi(y))dy) \tag{2.16}$$

where u(x) is a positive function, continuous and non-decreasing in x. Hence, by assumption, (2.16) implies

$$\phi(x) \le \prod_{\ell=1}^{k} E_\ell(x) \{u(x) + W(x,R(x))\} \tag{2.17}$$

where $E_\ell(x)$ is as defined earlier and R(x) is a solution of the integral equation (2.14) with f(x) replaced by u(x). Set $P(x) = \prod_{\ell=1}^{k} E_\ell(x)$; then P(x) is a positive function and so (2.17) becomes

$$\phi(x) \le P(x)\{f(x) + W(x,R(x))\} + P(x)q(x)\int_{x^0}^{x} g_{k+1}(y)H_{k+1}(\phi(y))dy.$$

By assumptions on q, P, f, W, and H_{k+1}, we have

$$\frac{\phi(x)}{P(x)[f(x) + W(x,R(x)]} \le 1 + q(x)\int_{x^0}^{x} g_{k+1}(y)H_{k+1}\left(\frac{\phi(y)}{P(y)[f(y) + W(y,R(y))]}\right) P(y)dy$$

$$\le q(x)\left[1 + \int_{x^0}^{x} P(y)g_{k+1}(y)H_{k+1}\left(\frac{\phi(y)}{P(y)[f(y) + W(y,R(y))]}\right)dy\right] . \tag{2.18}$$

Define $J : \Omega \to R$ such that

$$J(x) = 1 + \int_{x^0}^{x} g_{k+1}(s)P(s)H_{k+1}\left(\frac{\phi(s)}{P(s)(f(s) + W(s,R(s)))}\right)ds$$

and $\qquad J(x) = 1 \quad$ on $x_j = x_j^0$, $1 \le j \le n$.

Then $D_1D_2\ldots D_nJ(x) = g_{k+1}(x)P(x)H_{k+1}\left(\frac{\phi(x)}{P(x)\{f(x) + W(x,R(x))\}}\right)$

and, using (2.18) and the submultiplicative property of H_{k+1},

$$D_1D_2\ldots D_nJ(x) \le g_{k+1}(x)P(x)H_{k+1}(q(x))H_{k+1}(J(x)).$$

Hence

$$\frac{H_{k+1}(J(x))\cdot D_1D_2\ldots D_nJ(x)}{[H_{k+1}(J(x))]^2} \le g_{k+1}(x)P(x)H_{k+1}(q(x))$$

$$+ \frac{D_1D_2\ldots D_{n-1}J(x)\cdot D_nH_{k+1}(J(x))}{[H_{k+1}(J(x))]^2} ,$$

that is,

$$D_n\left(\frac{D_1D_2\cdots D_{n-1}J(x)}{H_{k+1}[J(x)]}\right) \le g_{k+1}(x)P(x)H_{k+1}(q(x)) . \tag{2.19}$$

Keeping $x_1,\ldots,x_{n-1}$ fixed in (2.19), setting $s_n = y_n$, and integrating with respect to y_n from x_n^0 to x_n, we have

$$\frac{D_1\cdots D_{n-1}J(x)}{H_{k+1}(J(x))} \le \int_{x_n^0}^{x_n} g_{k+1}(x_1\cdots x_{n-1},y_n)P(x_1\cdots x_{n-1},y_n)H_{k+1}(q(x_1\cdots x_{n-1},y_n))dy_n . \tag{2.20}$$

Set $\xi = (x_1,\ldots,x_{n-1},y_n)$ in (2.20), and use the same type of arguments to arrive at

$$D_{n-1}\left(\frac{D_1\cdots D_{n-1}J(x)}{H_{k+1}(J(x))}\right) \le \int_{x_n^0}^{x_n} g_{k+1}(\xi)P(\xi)H_{k+1}(q(\xi))dy_n. \tag{2.21}$$

Keeping $x_1\cdots x_{n-2}$ and y_n fixed, setting $x_{n-1} = y_{n-1}$, integrating (2.21) from x_{n-1}^0 to x_{n-1} with respect to y_{n-1}, and setting $\eta = (x_1,x_2,\ldots,x_{n-2})$, we have

$$\frac{D_1\cdots D_{n-2}J(x)}{H_{k+1}(J(x))} \le \int_{x_{n-1}^0}^{x_{n-1}}\int_{x^0}^{x_n} g_{k+1}(\eta,y_{n-1},y_n)P(\eta,y_{n-1},y_n)H_{k+1}(q(\eta,y_{n-1},y_n))dy_{n-1}dy_n.$$

Proceeding in this manner, we arrive at the inequality

$$\frac{D_1J(x)}{H_{k+1}(J(x))} \le \int_{x_2^0}^{x_2}\cdots\int_{x_n^0}^{x_n} g_{k+1}(x_1,y_2,\ldots,y_n)P(x_1,y_2,\ldots,y_n)H_{k+1}(q(x_1,y_2,\ldots,y_n))dy_2\cdots dy_n. \tag{2.22}$$

Using (2.8) and (2.22),

$$D_1G_{k+1}(J(x)) \le \int_{x_2^0}^{x_2}\cdots\int_{x_n^0}^{x_n} g_{k+1}(x_1,y_2,\ldots,y_n)P(x_1,y_2,\ldots,y_n)H_{k+1}(q(x_1,y_2,\ldots y_n))dy_2\cdots dy_n.$$

Finally, keeping $y_2,\ldots,y_n$ fixed, setting $x_1 = y_1$, and integrating with respect to y_1 from x_1^0 to x_1, we obtain

$$G_{k+1}(J(x)) \le G_{k+1}(1) + \int_{x^0}^{x} g_{k+1}(y)P(y)H_{k+1}(q(y))dy,$$

so that $J(x) \leq G_{k+1}^{-1}[G_{k+1}(1) + \int_{x^0}^{x} g_{k+1}(y)P(y)H_{k+1}(q(y))dy]$.

Consequently, by (2.18),

$$\frac{\phi(x)}{P(x)[f(x) + W(x,R(x))]} \leq q(x)J(x)$$
$$\leq q(x)G_{k+1}^{-1}[G_{k+1}(1) + \int_{x^0}^{x} g_{k+1}(y) \prod_{\ell=1}^{k} E_\ell(y)H_{k+1}(q(y))dy]$$
$$= E_{k+1}(x),$$

that is,

$$\phi(x) \leq \prod_{\ell=1}^{k} E_\ell(x) \cdot E_{k+1}(x)\{f(x) + W(x,R(x))\}$$
$$= \prod_{\ell=1}^{k+1} E_\ell(x)\{f(x) + W(x,R(x))\}$$

where $R(x)$ is a solution of

$$R(x) = \int_{x^0}^{x} K(x,y, \prod_{\ell=1}^{k} E_\ell(y)\{u(y) + W(y,R(y))\}\, dy.$$

Now $\quad u(x) = f(x) + q(x)\int_{x^0}^{x} g_{k+1}(y)H_{k+1}(\phi(y))dy,$

so that

$$u(x) + W(x,R(x)) = f(x) + W(x,R(x)) + q(x)\int_{x^0}^{x} g_{k+1}(y)H_{k+1}(\phi(y))dy.$$

Using inequality (2.17) and a property of H_{k+1},

$$u(x) + W(x,R(x) \leq f(x) + W(x,R(x)) + q(x)\int_{x^0}^{x} g_{k+1}(y)H_{k+1}\left(\prod_{\ell=1}^{k} E_\ell(y)\{u(y) + W(y,R(y))\}\right)dy$$
$$\leq f(x) + W(x,R(x)) + q(x)\int_{x^0}^{x} g_{k+1}(y) \frac{\prod_{\ell=1}^{k} E_\ell(y)}{\prod_{\ell=1}^{k} E_\ell(y)} H_{k+1}\left(\prod_{\ell=1}^{k} E_\ell(y)\{u(y) + W(y,R(y))\}\right)dy$$

$$\leq f(x) + W(x,R(x)) + q(x)\int_{x^0}^{x} g_{k+1}(y) \prod_{\ell=1}^{k} E_\ell(y)H_{k+1}(u(y) + W(y,R(y)))dy.$$

Set $m(x) = u(x) + W(x,R(x))$, $n(x) = f(x) + W(x,R(x))$, and apply Theorem A to the above inequality; then we obtain

$$m(x) \leq n(x)q(x)[G_{k+1}^{-1}\{G_{k+1}(1) + \int_{x^0}^{x} g_{k+1}(y)\prod_{\ell=1}^{k} E_\ell(y)H_{k+1}(q(y))dy\}]$$

$$\leq E_{k+1}(x)\{f(x) + W(x,R(x)\} .$$

Hence, $$u(x) + W(x,R(x)) \leq E_{k+1}(x)\{f(x) + W(x,R(x))\}$$

so that $$\prod_{\ell=1}^{k} E_\ell(y)\{u(y) + W(y,R(y))\} \leq \prod_{\ell=1}^{k+1} E_\ell(y)\{f(y) + W(y,R(y))\}$$

and $$R(x) \leq \int_{x^0}^{x} K(x,y, \prod_{\ell=1}^{k+1} E_\ell(y)\{f(y) + W(y,R(y))\})dy$$

by the assumption on K.

Define $V_0(x) = R(x)$ and, for $j = 1,2,...,$

$$V_j(x) = \int_{x^0}^{x} K(x,y, \prod_{\ell=1}^{k+1} E_\ell(y)\{f(y) + W(y,V_{j-1}(y))\})dy.$$

Then $$V_1(x) = \int_{x^0}^{x} K(x,y, \prod_{\ell=1}^{k+1} E_\ell(y)\{f(y) + W(y,R(y))\})dy \geq R(x).$$

Hence, by the assumptions on K and W, we have

$$R(x) \leq V_1(x) \leq V_2(x) \leq \dots \leq V_J(x) \leq \dots$$

and, by the uniform Lipschitz continuity of K in the last variable and the Arzela's theorem, the sequence $\{V_j(x)\}$ converges to a unique solution V(x) of the integral equation

$$V(x) = \int_{x^0}^{x} K(x,y, \prod_{\ell=1}^{k+1} E_\ell(y)\{f(y) + W(y,V(y))\})dy \qquad (2.23)$$

and $R(x) \leq V(x)$ existing on Ω.

Thus, since W is nondecreasing in the last variable,

$$\phi(x) \leq \{f(x) + W(x,R(x)\}\prod_{\ell=1}^{k+1} E_\ell(x) \leq \{f(x) + W(x,V(x)\}\prod_{\ell=1}^{k+1} E_\ell(x)$$

where V(x) is a solution of (2.23). We have shown that, if (2.6) implies (2.7) for m = k, then (2.6) implies (2.7) for m = k+1, so that the proof is complete by the induction hypothesis.

REMARK: If q(x) = 1, n = 1, W(x,u) ≡ 0 for all x,u, then our result reduces to Theorem 1 of [8]. When m = 1, we have Theorem 3 of [23], so that our result generalizes the result of Pachpatte.

3. APPLICATION.

In this section we shall demonstrate the usefulness of the inequality established in section 2 by obtaining pointwise bounds on the solutions of a certain class of non-linear equations in n-independent variables. We consider the non-linear hyperbolic partial integrodifferential equation

$$\frac{\partial^n \phi(x)}{\partial x_1 \partial x_2 \ldots \partial x_n} = F\left(x, \phi(x), \int_{x^0}^{x} K(x,y,\phi(y))dy\right) + G(x,\phi(x)) \tag{3.1}$$

where $F \in C(\Omega \times R \times R, R)$ and $G \in C(\Omega \times R, R)$. With suitable boundary conditions the solution of (3.1) is of the form

$$\phi(x) = h(x) + \int_{x^0}^{x} F\left(s, \phi(s), \int_{x^0}^{x} K(s,y,\phi(y))dy\right)ds + \int_{x^0}^{x} G(y,\phi(y))dy \tag{3.2}$$

We shall assume the following conditions:

(H_7) There exists a continuous function $B : \Omega \times R^+ \to R^+$ with B nondecreasing in the second variable such that

$$|G(y,\phi(y))| \le B(y, |\phi(y)|). \tag{3.3}$$

(H_8) There exists a function $f : \Omega \to R$ satisfying (H_1) such that $|h(x)| \le f(x)$, for all $x \in \Omega$.

(H_9) There exists a function $g : \Omega \to R^+$ satisfying the assumption (H_2) such that for $s \in \Omega$,

$$|F(s,u,v)| \le g(s)[|u| + |v|] \tag{3.4}$$

(H_{10}) There exist functions $\omega : \Omega \times \Omega \to R$ and $H : R^+ \to R^+$ such that

(i) $\omega(s,y)$ is defined and continuous for $s \ge y \ge x^0$

(ii) $\omega(s,s) \le h_1(s)$, $D_j\omega(s,s_1,s_2,\ldots,s_{j-1},y_j,\ldots,y_n) = 0$, $j = 2,3,\ldots,n$, $D_1D_2\ldots D_i\omega(s,y_1,y_2,\ldots,y_i,s_{i+1},y_{i+2},\ldots,y_n) = 0$, $i = 1,2,\ldots,n-1$ and $D_1D_2,,,D_n\omega(s,y) \le p(s)h_2(y)$ where h_1, p, and h_2 are continuous functions and non-negative on Ω with $D_\ell = \frac{\partial}{\partial s_\ell}$ $1 \le \ell \le n$.

(iii) H satisfies assumption (H_6) with $H(1) = 1$

(iv) $|K(s,y,\phi(y))| \le \omega(s,y)H(|\phi(y)|)$ (3.5)

REMARK: It is easy to see that the function

$\omega(s,y) = \prod_{\ell=1}^{n} (s_\ell - y_\ell)p(s)h_2(y) + C$, where C is constant, satisfies (i), (ii), and (iii) if $C \le h_1(s)$ for all $s \in \Omega$.

The following Lemma, which is a standard result in calculus of several variables, shall be used in obtaining pointwise bounds on the solution of (3.1).

LEMMA 1. Let $G(s) = \int_{x^0}^{s} \omega(s,y)H(\phi(y))dy$ with $x^0 = (x_1^0,\dots,x_n^0)$, $y = (y_1,y_2,\dots,y_n)$ and $s = (s_1,s_2,\dots,s_n) \in \Omega$ with $x^0 < y < s$, and $D_i = \frac{\partial}{\partial s_i}$ $i = 1,2,\dots,n$. Assume, also, that

$D_j\omega(s,s_1,s_2,\dots,s_{j-1},y_j,\dots,y_n) = 0$ for $j = 2,3,\dots,n$ and

$D_1D_2\dots D_k\omega(s,y_1,y_2,\dots,y_k,s_{k+1},y_{k+2},\dots,y_n) = 0$, $k = 1,2,\dots,n-1$.

Then, $D_1D_2\dots D_nG(s) = \omega(s,s)H(\phi(s)) + \int_{x^0}^{s} D_1D_2\dots D_n\omega(s,y)H(\phi(y))dy$.

We now compute the pointwise bounds of the integral equation (3.2) taking into account the assumptions (H_7) - (H_{10}).

Taking the bounds in (3.2) and using (3.3), (3.4), and (3.5), we obtain

$$|\phi(x)| \le |h(x)| + \int_{x^0}^{x}|G(y,\phi(y))|dy + \int_{x^0}^{x}\left|F\left(s,\phi(s),\int_{x^0}^{s} K(s,y,\phi(y))dy\right)\right|ds$$

$$\le f(x) + \int_{x^0}^{x} B(y,|\phi(y)|)dy + \int_{x^0}^{x} g(s)|\phi(s)|ds + \int_{x^0}^{x} g(s)\left|\int_{x^0}^{s} K(s,y,\phi(y))dy\right|ds$$

$$\le f(x) + \int_{x^0}^{x} B(y,|\phi(y)|)dy + \int_{x^0}^{x} g(s)|\phi(s)|ds + \int_{x^0}^{x} g(s)\left(\int_{x^0}^{s} \omega(s,y)H(|\phi(y)|)dy\right)ds.$$

In view of hypothesis (H_{10}) and Lemma 1, if $R(s) = \int_{x^0}^{s} \omega(s,y)H(|\phi(y)|)dy$, then

$$D_1\dots D_nR(s) = \omega(s,s)H(|\phi(s)|) + \int_{x^0}^{s} D_1D_2\dots D_n\omega(s,y)H(|\phi(y)|)dy$$

$$\le h_1(s)H(|\phi(s)|) + \int_{x^0}^{s} p(s)h_2(y)H(|\phi(y)|)dy$$

$$= h_1(s)H(|\phi(s)|) + p(s)\int_{x^0}^{s} h_2(y)H(|\phi(y)|)dy.$$

Upon integrating from x^0 to s, we obtain

$$R(s) \le \int_{x^0}^{s} h_1(u)H(|\phi(u)|)du + \int_{x^0}^{s} p(u)\left(\int_{x^0}^{u} h_2(y)H(|\phi(y)|)dy\right)du.$$

Hence

$$|\phi(x)| \le f(x) + \int_{x^0}^{x} g(s)|\phi(s)|ds + \int_{x^0}^{x} g(s)\left(\int_{x^0}^{s} h_1(u)H(|\phi(u)|)du\right)ds$$

$$+ \int_{x^0}^{x} g(s)\left(\int_{x^0}^{s} p(u)\left(\int_{x^0}^{u} h_2(y)H(|\phi(y)|)dy\right)du\right)ds$$

$$+ \int_{x^0}^{x} B(y,|\phi(y)|)dy.$$

We now use Theorem 1 with $g_i = g \quad j = 1,2,3, \quad q = 1, \quad H_1(|\phi(s)|) = |\phi(s)|$,

$H_2(|\phi(s)|) = \int_{x^0}^{s} h_1(u)H(|\phi(u)|)du, \quad H_3(|\phi(s)|) = \int_{x^0}^{s} p(u)\left(\int_{x^0}^{u} h_2(y)H(|\phi(y)|)dy\right)du,$

$K(x,y,u) = B(y,u), \ W(s,z) = z;$ then $m = 3$ and we have

$$|\phi(x)| \le \{f(x) + r(x)\} \prod_{\ell=1}^{3} E_i(x) \tag{3.6}$$

$$= \{f(x) + r(x)\}E_1(x)\cdot E_2(x)\cdot E_3(x)$$

where $r(x)$ is a solution of the equation

$$v(x) = \int_{x^0}^{x} B(y,E_1(y)E_2(y)E_3(y)\{f(y) + v(y)\})dy$$

and

$$E_1(x) = G_1^{-1}[G_1(1) + \int_{x^0}^{x} g(s)ds], \quad G_1(1) + \int_{x^0}^{x} g(s)ds \in \mathrm{Dom}(G_1^{-1})$$

$$E_2(x) = G_2^{-1}[G_2(1) + \int_{x^0}^{x} g(s)E_1(s)H_2(1)ds]$$

$$= G_2^{-1}[G_2(1) + \int_{x^0}^{x} g(s)E_1(s)\left(\int_{x^0}^{x} h_1(u)du\right)ds]$$

$$E_3(x) = G_3^{-1}[G_3(1) + \int_{x^0}^{x} g(s)E_1(s)E_2(s)\left(\int_{x^0}^{s} p(u)\left(\int_{x^0}^{u} h_2(y)dy\right)du\right)ds].$$

It is clear that we can compute the pointwise bounds of the solution $\phi(x)$ of the integral equation (3.2) as in (3.6).

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REFERENCES

1. BECKENBACH, E.F. and BELLMAN, R. Inequalities, Springer/Verlag, Berlin 1961.

2. BEESACK, P.R. Gronwall Inequalities, Carleton Mathematical Lecture Notes no. 1, 1975.

3. BIHARI, I. A generalization of a Lemma of Bellman and its application to uniqueness problems of differential equations, Acta Math. Acad. Sci. Hung. 7 (1956), 81-94.

4. CHANDRA, J. and DAVIS, P.W. Linear generalizations of Gronwall's inequality, Proc. Amer. Math. Soc. 60 (1976) 157-160.

5. CHANDRA, J. and FLEISHMAN, B.A. On a generalization of the Gronwall-Bellman Lemma in partially ordered Banach spaces, J. Math. Anal. Appl. 31 (1970), 668-681.

6. CHU, S.C. and METCALF, F.T. On Gronwall's inequality, Proc. Amer. Math. Soc. 18 (1967), 437-440.

7. DHONGADE, U.D. and DEO, S.G. Some generalizations of Bellman-Bihari integral inequalities, J. Math. Anal. Appl. 44 (1973), 218-226.

8. DHONGADE, U.D. and DEO, S.G. A non-linear generalization of Bihari's inequality, Proc. Amer. Math. Soc. 54 (1976), 211-216.

9. PACHPATTE, B.G. On some integral inequalities similar to Bellman-Bihari inequalites, J. Math. Anal. Appl. 49 (1975), 794-802.

10. PACHPATTE, B.G. A note on Gronwall-Bellman inequality, J. Math. Anal. Appl. 44 (1973), 758-762.

11. BONDGE, B.K. and PACHPATTE, B.G. On some fundamental integral inequalities in two independent variables, J. Math. Anal. Appl. 72 (1979), 533-544.

12. BONDGE, B.K. and PACHPATTE, B.G. On two independent variable generalization of certain integral inequalities, Tamkang J. Math. 11 (1980), no. 1, 37-47.

13. BONDGE, B.K. and PACHPATTE, B.G. On nonlinear integral inequalities of the Wendroff type, J. Math. Anal. Appl. 70 (1979), 161-169.

14. BONDGE, B.K., PACHPATTE, B.G. and WALTER, W. On generalized Wendroff-type inequalities and their applications, Non-linear Anal. Theory Methods and Appl. 4 (1979), 491-495.

15. PACHPATTE, B.G. On Wendroff type partial integral inequalities, Tamkang J. Math 10 (1979), 141-150.

16. PACHPATTE, B.G. On some fundamental partial integral inequalities, J. Math. Anal. Appl. 73 (1980), 238-251.

17. PACHPATTE, B.G. On some new integrodifferential inequalities of the Wendroff type, J. Math. Anal. Appl. 73 (1980), 491-500.

18. PACHPATTE, B.G. On some new integral inequalities for non self-adjoint hyperbolic partial integrodifferential equations, J. Math. Anal. Appl. 76 (1980), 58-71.

19. RASMUSSEN, D.L. Gronwall's inequality for functions of two independent variables, J. Math. Anal. Appl. 55 (1976), 407-417.

20. SNOW, D.R. Gronwall's inequality for system of partial differential equation in two independent variables, Proc. Amer. Math. Soc. 33 (1972), 46-54.

21. CONLAN, J. and DIAZ, J.B. Existence of solutions of an n^{th} order hyperbolic partial differential equations, Contributions to Differential Equations, 2 (1963), 277-289.

22. HEADLEY, V.B. A multidimensional non-linear Gronwall inequality, J. Math. Anal. Appl. 47 (1974), 250-255.

23. PACHPATTE, B.G. On some partial integral inequalities in n-independent variables, J. Math. Anal. Appl. 79 (1981), 256-272.

24. SINGARE, V.M. and PACHPATTE, B.G. Lower bounds on some integral inequalities in n-independent variables, Indian J. Pure and Appl. Math. 12 (1981), no. 3, 318-331.

25. SHIH, MAU-HSIANG. and YEH, CHEH-CHIH. Some integral inequalities in n-independent variables, J. Math. Anal. Appl. 84 (1981), 569-583.

26. YEH, CHEH-CHIH. Wendroff's inequality in n-independent variables, J. Math. Anal. Appl. 78 (1980), 78-87.

27. YOUNG, E.C. Gronwall's inequality in n-independent variables, Proc. Amer. Math. Soc. 41 (1973), 241-244.

28. ZAHARIEV A.I. and BAINOV, D.D. A note on Bellman-Gronwall's inequality, J. Math. Anal. Appl. 80 (1981), 147-149.