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DAVID J. FIELDHOUSE

Department of Mathematics and Statistics
University of Guelph
Guelph, Ontario N1G 2W1 CANADA

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ABSTRACT. Orzech [1] has shown that every surjective endomorphism of a noetherian module is an isomorphism. Here we prove analogous results for injective endomorphisms of noetherian injective modules, and the duals of these results. We prove that every injective endomorphism, with large image, of a module with the descending chain condition on large submodules is an isomorphism, which dualizes a result of Varadarajan [2]. Finally we prove the following result and its dual: if p is any radical then every surjective endomorphism of a module M, with kernel contained in pM, is an isomorphism, provided that every surjective endomorphism of pM is an isomorphism.

KEY WORDS AND PHRASES. Injective endomorphism, surjective endomorphism, ascending chain condition (ACC), descending chain condition (DCC), artinian module, noetherian module, injective module, projective module, injective hull, projective cover, small submodule, large submodule, preradical, radical, idempotent preradical.
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1. INTRODUCTION.

Orzech [1] has shown that every surjective endomorphism of a noetherian module is an isomorphism. Here we prove analogous results for injective endomorphisms of noetherian injective modules, and the duals of these results. We prove that every injective endomorphism, with large image, of a module with the descending chain condition on large submodules is an isomorphism, which dualizes a result of Varadarajan [2]. Finally we prove the following result and its dual: if p is any radical then every surjective endomorphism of a module M, with kernel contained in pM, is an isomorphism, provided that every surjective endomorphism of pM is an isomorphism.

2. CONVENTIONS, NOTATION, AND TERMINOLOGY.

Unless otherwise stated, we use the following conventions, notation, and terminology.

All rings are associative, but not necessarily commutative. Every ring has a multiplicative identity element, denoted by 1, which is preserved by ring homomorphisms, inherited by subrings, and acts as the identity operator on modules.

We use the word map for module homomorphism. Maps are written on the side opposite to that of the scalars. Thus the order of writing map compositions depends on the side of the module.

If M and N are R-modules we usually write Hom(M,N) for $\mathrm{Hom}_R(M,N)$ when no confusion can arise.

The symbols $<$ and $>$ will be used to denote proper set theoretical inclusion and containment, respectively, as well as the usual order relationships. The symbols $\leq$ and $\geq$, respectively, are used for the preceeding if equality can occur.

We recall that a module is noetherian iff it satisfies the ascending chain condition (ACC) for submodules, and artinian iff it satisfies the descending chain condition (DCC) for submodules.

A submodule L of a module M is defined to be large (or essential) iff it has a non-zero intersection with every non-zero submodule of M. A map is large iff its image is a large submodule. It is easy to verify that the product of large injective maps is large and that under a surjective map preimages of large submodules are large.

Dually, a submodule S of a module M is defined to be small (or superfluous) iff whenever $S + M' = M$ for a submodule M' of M then we must have $M' = M$. A map is small iff its kernel is a small submodule. It is easy to verify that the product of small surjective maps is small and that small submodules are small in overmodules.

In the following we shall state a number of results and their duals, but usually only give the proof of one.

3. MAIN RESULTS.

Let us fix a right R-module M and a submodule M', and denote the factor module M/M' by M''.

THEOREM 1.

(1) If M'' is artinian then every injective map in $\mathrm{Hom}(M,M'')$ is an isomorphism.

(2) If M' is noetherian then every surjective map in $\mathrm{Hom}(M',M)$ is an isomorphism.

PROOF. Part (2) is basically a Theorem of Orzech [1]. For the sake of completeness we shall prove its dual, part (1).

Let f be an injective map in $\mathrm{Hom}(M,M'')$. We define the following descending chain of submodules of M: let $M_0 = M$, and for $n > 0$, M_{n+1} is defined by $M_{n+1}/M' = fM_n$.

One verifies readily that the M_n form a descending chain of submodules of M, which gives rise to the descending chain M_n/M' of submodules of M/M', which must terminate since M'' is artinian. Let n be the least integer such that $M_{n+1}/M' = M_n/M'$. If $n = 0$ then the map f is surjective, and hence an isomorphism. Otherwise, if $n > 0$ then $fM_n = fM_{n-1}$, which is impossible since $M_n < M_{n-1}$ and the map f is injective.

We now turn to the dual problems.

THEOREM 2.

(1) If M'' is an artinian module and M is a projective module then every surjective map in $\mathrm{Hom}(M'',M)$ is an isomorphism.

(2) If M' is a noetherian module and M is an injective module then every injective map in $\mathrm{Hom}(M',M)$ is an isomorphism.

PROOF. We shall prove only part (1) and leave the proof of this and all subsequent dual proofs to the reader.

Let f be a surjective map in $\mathrm{Hom}(M'',M)$. Since M is projective this map splits, so that $fg = 1_M$, the identity map on M, with g an injective map in $\mathrm{Hom}(M,M'')$. The preceeding Theorem implies that g is an isomorphism, and hence f is too.

COROLLARY.

(1) If M is an artinian projective module then every surjective map in $\mathrm{Hom}(M'',M)$ is an isomorphism.

(2) If M is a noetherian injective module then every injective map in $\mathrm{Hom}(M,M')$ is an isomorphism.

THEOREM 3.

(1) If M has an artinian projective cover then every surjective map in $\mathrm{Hom}(M,M)$ is an isomorphism.

(2) If M has a noetherian injective hull then every injective map in $\mathrm{Hom}(M,M)$ is an isomorphism.

PROOF.

(1) Let P be an artinian projective cover of M, with corresponding kernel K. Any surjective map f in Hom(M,M) lifts to a map g in Hom(P,P) which must be surjective since P is a cover. Since P is artinian and projective g must be an isomorphism, which means that its restriction to K is injective. But K is artinian since P is, and therefore the restriction must be an isomorphism. Hence the map f is an isomorphism.

(2) has a dual proof.

The following theorem gives the dual of and a slight generalization of a result of Varadarajan [2]. The proof we give, and its dual, differ from his.

THEOREM 4.

(1) If M" has the descending chain condition on large submodules then every injective map in Hom(M,M") with large image is an isomorphism.

(2) If M′ has the ascending chain condition on small submodules then every surjective map in Hom(M′,M) with small kernel is an isomorphism.

PROOF.

(1) Let f be an injective map in Hom(M,M"). As before we define a descending chain of submodules: $M_0 = M$ and for $n > 0$, $M_{n+1}/M' = fM_n$.

In order to use the proof given above, we must verify that we now have a descending chain of large submodules of M". We do this by induction. By assumption, for $n = 0$ we have $M_1/M' = fM$ large in M". Assume now that M_n/M' is large in M/M'. Since under a surjection preimages of large submodules are large, we have M_n large in M. We need only show that M_{n+1}/M' is large in M_n/M' and then use the transitivity of largeness. Let m" be a non-zero element of $M_n/M' = fM_{n-1}$. Since f is injective there exists a unique (non-zero) element m in M_{n-1} such that fm = m". Since M_n is large in M there exists an element r in R such that rm is a non-zero element of M_n. Since f is injective the element f(rm) = rm" is a non-zero element of M_{n+1}, which concludes the proof.

(2) The preceeding proof can be dualized.

COROLLARY.

(1) If M has the descending chain condition on large submodules then every injective map in Hom(M,M") with large image is an isomorphism.

(2) If M has the ascending chain condition on small submodules then every surjective map in Hom(M′,M) with small kernel is an isomorphism.

PROOF.

(1) Under a surjective map preimages of large submodules are large.

(2) Small submodules are small in overmodules.

4. PRERADICALS.

For preradicals we generally use the notation and terminology of the book of Stenstrom [3], with some minor modifications. We use the letter p to denote a preradical, and 0 and 1 to denote the zero and identity functors, respectively.

A preradical is defined to be a subfunctor of the identity functor.

Each preradical p defines an ascending chain of preradicals p_a, indexed by the ordinals, as follows: for the ordinal a = 0: $p_0M = 0$; for non-limit ordinals a > 0: $p_aM/p_{a-1}M = p(M/p_{a-1}M)$; and for limit ordinals a: $p_aM = \sum p_bM$ with the sum taken over all ordinals b < a.

If we let $p' = \sum p_a$ then we have an ascending chain of preradicals:

$$0 = p_0 \leq p = p_1 \leq p_2 \leq \dots \leq p_a \leq \dots \leq p' \leq 1.$$

Each preradical p also defines a descending chain of preradicals p^b, indexed by the ordinals, as follows: for the ordinal b = 0: $p^0M = M$; for non-limit ordinals b > 0: $p^bM = p(p^{b-1}M)$; and for limit ordinals b, p^bM is the intersection of all p^aM with the intersection taken over all a < b.

If we let p" denote the intersection of the p^b then we have a descending chain of preradicals:

$$1 = p^0 \geq p = p^1 \geq p^2 \geq \dots \geq p^b \geq \dots \geq p'' \geq 0.$$

A preradical p is radical iff p = p', or equivalently, p(M/pM) = 0 for all modules M. For any preradical p we have (p')' = p' and hence p' is radical.

A preradical p is idempotent iff p(pM) = pM for all modules M. It is easy to verify that for any preradical p the preradical p" is idempotent.

For any module M and ordinals a and b let $M_a = p_aM$ and $M^b = p^bM$. This defines a corresponding ascending chain of submodules:

$$0 = M_0 \leq M_1 \leq M_2 \leq \dots \leq M_a \leq \dots \leq M$$

and a corresponding descending chain of submodules:

$$M = M^0 \geq M^1 \geq M^2 \geq \dots \geq M^b \geq \dots \geq 0.$$

LEMMA. Let M and N be right R-modules and p be any preradical.
(1) If $M_a = M_{a+1}$ and f is a surjective map in Hom(M,N) with kernel contained in M_a then the naturally induced map f_a in Hom(M_a,N_a) is surjective.
(2) If $M^b = M^{b+1}$ and f is an injective map in Hom(N,M) with image containing M^b then the naturally induced map f^b in Hom(N/N^b,M/M^b) is injective.

PROOF.

(1) Since p_a is a preradical the image under f of M_a is contained in N_a. Since the kernel of f is contained in M_a there is a canonical map from N to M/M_a, under which the image of N_a is contained in $p_a(M/M_a)$, which is zero since $M_a = M_{a+1}$. This implies that N_a equals the image of M_a under f, i.e. f_a is surjective.

(2) The proof is dual.

THEOREM 5. Let p be any preradical.
(1) If $p_a = p_{a+1}$ and every surjective map from a submodule of M_a to M_a itself is an isomorphism, then every surjective map in Hom(M′,M) with kernel contained in M'_a is an isomorphism.
(2) If $p^b = p^{b+1}$ and every injective map from M^b itself to a factor module of M^b is an isomorphism, then every injective map in Hom(M,M") with image containing M''^b is an isomorphism.

PROOF.
(1) If f is a surjective map in Hom(M′,M) then the induced map f_a in Hom(M'_a,M_a) is surjective and hence an isomorphism since M'_a is a submodule of M_a. Since the kernel of f is contained in M'_a it must be zero.
(2) The proof is dual.

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