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# A UNIFIED APPROACH TO RADIUS OF CONVEXITY PROBLEMS FOR CERTAIN CLASSES OF UNIVALENT ANALYTIC FUNCTIONS

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ABSTRACT. We consider functions f analytic in the unit disc and assume the power series representation of the form

$$f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots$$

where $a_{n+1}$ is fixed throughout. We provide a unified approach to radius convexity problems for different subclasses of univalent analytic functions. Numerous earlier estimates concerning the radius of convexity such as those involving fixed second coefficient, n initial gaps, n+1 symmetric gaps, etc. are discussed. It is shown that several known results follow as special cases of those presented in this paper.

*KEY WORDS AND PHRASES. Radius of convexity, univalent functions, starlike functions of order* $\alpha$ *and type* $\beta$, $(n+1)$*-fold symmetric functions, functions with positive real part.*

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## 1. INTRODUCTION.

This paper is directed to mathematical specialists familiar with conformal mapping and univalent function theory. Let P denote the class of functions of the form

$$p(z) = 1 + b_1 z + b_2 z^2 + \dots \tag{1.1}$$

which are analytic and satisfy Re $p(z) > 0$ for $z \in E = \{z: |z| < 1\}$. Let $P(\alpha,\beta)$ denote the class of functions of the form (1.1) which are analytic in $E$ and satisfy the inequality

$$\left| \frac{p(z) - 1}{2\beta(p(z) - \alpha) - (p(z) - 1)} \right| < 1, \tag{1.2}$$

for some $\alpha, \beta (0 \le \alpha < 1,\ 0 < \beta \le 1)$ and for all $z \in \Delta$. It is easily seen that $P(0,1) \equiv P$. In fact, for different values of $\alpha, \beta$, the class $P(\alpha,\beta)$ yields a number of other subclasses of P that have been studied by various workers. The advantage of studying different aspects of P and its subclasses is that the results obtained for these subclasses can be successfully applied to obtain estimates for various subclasses of starlike functions and for functions whose derivatives have a positive real part in $\Delta$. For example, Tepper [1] considered starlike functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots , \tag{1.3}$$

with fixed second coefficient and used the above approach with $p \in P$ to obtain a sharpened radius of convexity result for this class. This also led him to an improvement of an old estimate in support of Schild's conjecture. Later, these results were generalized by McCarty [2] to starlike functions of order $\alpha$ by considering $p \in P(\alpha,1)$. The authors in [3] considered the problem in a very general setting by considering $p \in P(\alpha,\beta)$.

Recently, the authors [4] considered starlike functions of the form

$$f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots , \tag{1.4}$$

with the coefficient $a_{n+1}$ as fixed and obtained results that led to further improvement of Tepper's results. Here it may be pointed out that it was, perhaps, for the first time, that this problem was solved by taking the general $(n + 1)$th coefficient fixed. Although the results of this paper did lead to sharpening of earlier known results for the whole class of starlike functions, because of taking $p \in P$, they could not be applied to a number of various interesting subclasses of starlike functions that have been studied by numerous workers from time to time.

The aim of the present paper is to provide a unified approach for studying radius of convexity problems for different subclasses of univalent analytic functions. This is done by considering functions of the form (1.4) having fixed coefficient $a_{n+1}$ throughout and by considering the corresponding $p \in P(\alpha,\beta)$. This yields not only second fixed coefficient results for $n = 1$ obtained earlier in [2-3] etc., but also leads to refinement of these results for $n > 1$. Whereas the substitution of the corresponding coefficient estimate for $|a_{n+1}|$ leads to radius of convexity results for functions with n initial gaps, putting $a_{n+1} = 0$ gives results for functions with (n+1)-symmetric gaps. Thus, our approach solves radius of convexity problems for all the different situations that the workers in this field have been considering. Since we are attacking the problem in a general setting by involving the parameters $\alpha,\beta$, the different subclasses are also being covered.

We introduce the following subclasses: Let, for $0 \le \alpha < 1$, $0 \le \beta < 1$, $n \ge 1$,

$$P_n(\alpha,\beta) = \{p : p(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \dots \text{ and } p(z) \text{ satisfies (1.2) for all } z \in \Delta\}$$

$$P_n(b;\alpha,\beta) = \{p \in P_n(\alpha,\beta) : b_n = 2\beta(1-\alpha)b,\ 0 \le b \le 1\}.$$

## 2. PRELIMINARY LEMMAS.

Let B denote the class of functions w analytic in $\Delta$ which satisfy (i) $w(0) = 0$ and (ii) $|w(z)| < 1$ for $z \in \Delta$. We need the following lemmas.

LEMMA 1 [5]. If $w \in B$, then for all $z \in \Delta$

$$|zw'(z) - w(z)| \le \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}. \tag{2.1}$$

LEMMA 2. Let $w \in B$. Then we have

$$\operatorname{Re}\left\{ \frac{z^n w'(z) + (n-1)z^{n-1}w(z)}{(1 + sz^{n-1}w(z))(1 + tz^{n-1}w(z))} \right\} \le -\frac{n}{(s-t)^2} \operatorname{Re}\left\{sp(z) + \frac{t}{p(z)} - s - t\right\} + \frac{r^{2n}|sp(z) - t|^2 - |1 - p(z)|^2}{(s-t)^2 r^{n-1}(1 - r^2)|p(z)|}, \tag{2.2}$$

where $p(z) = (1 + tz^{n-1}w(z))/(1 + sz^{n-1}w(z))$, $|z| = r$, $-1 \le t < s \le 1$ and $n \ge 1$ is a fixed integer.

Using the estimate (2.1), the lemma follows easily. Hence we omit the proof.

LEMMA 3. If $p(z) = (1 + tz^{n-1}w(z))/(1 + sz^{n-1}w(z))$ and $w \in b$, then for each $b \in [0,1]$ and s,t satisfying $-1 \le t < s \le 1$, p(z) lies in the disc

$$\Delta(z) \equiv \{\zeta : |\zeta - A_{b,n}| \le D_{b,n}\},$$

where

$$A_{b,n} = \frac{(1 + br)^2 - str^{2n}(b + r)^2}{(1 + br)^2 - s^2 r^{2n}(b + r)^2}, \qquad D_{b,n} = \frac{(s - t)r^n(b + r)(1 + br)}{(1 + br)^2 - s^2 r^{2n}(b + r)^2}$$

and $r = |z| < 1$.

PROOF. Since $p(z) = (1 + tz^{n-1}w(z))/(1 + sz^{n-1}w(z))$, we have

$$w(z) = \frac{1 - p(z)}{z^{n-1}(sp(z) - t)} = -[bz + \dots] = -z\psi(z)$$

where $\psi$ is analytic and $|\psi(z)| \le 1$ for $z \in \Delta$ with $\psi(0) = b$. Now, since $(\psi(z)-b) \div (1-b\psi(z))$ is subordinate to z, it follows that $\psi(z)$ is subordinate to $(z + b)/(1 + bz)$ and so

$$\left| \frac{1 - p(z)}{sp(z) - t} \right| \le |z|^n \frac{(|z| + b)}{(1 + b|z|)} . \tag{2.3}$$

Taking $p(z) = \xi + i\eta$, (2.3) gives

$$\left|\xi + i\eta - \frac{(1 + br)^2 - str^{2n}(b + r)^2}{(1 + br)^2 - s^2r^{2n}(b + r)^2}\right| \le \frac{(s - t)r^n(b + r)(1 + br)}{(1 + br)^2 - s^2r^{2n}(b + r)^2} .$$

Hence the lemma follows.

LEMMA 4. If $p(z) = (1 + tz^{n-1}w(z))/(1 + sz^{n-1}w(z))$ and $w \in B$, then for $|z| = r$, we have

$$\mathrm{Re}\,\{qp(z) + \frac{nt}{p(z)}\} - \frac{r^{2n}|sp(z) - t|^2 - |1 - p(z)|^2}{r^{n-1}(1 - r^2)|p(z)|}$$

$$\ge \begin{cases} \dfrac{2}{r^{n-1}(1-r^2)}[\sqrt{(1+qr^{n-1}(1-r^2)-s^2r^{2n})(1+ntr^{n-1}(1-r^2)-t^2r^{2n})}-(1-str^{2n})] & \text{if } R_{b,n} \le R^*_n \\ \dfrac{W}{W^*} & \text{if } R_{b,n} \ge R^*_n \end{cases} \tag{2.4}$$

where

$$W = (q + nt)(1 + br)^2 + 2bt(q + ns)r^n(1 + r^2) + (-(1 - b^2)(s - t) + 2t(q + ns)(1 + b^2))r^{n+1} + (t^2q + s^2nt)r^{2n}(b + r)^2, \tag{2.5}$$

$$W^* = (1 + br + bsr^n + sr^{n+1})(1 + br + btr^n + tr^{n+1}), \tag{2.6}$$

$R^{*2}_n = (1 + ntr^{n-1}(1 - r^2) - t^2r^{2n})/(1 + qr^{n-1}(1 - r^2) - s^2r^{2n})$ and $R_{b,n} = A_{b,n} - D_{b,n}$ where $A_{b,n}$, $D_{b,n}$ are defined as in Lemma 3 and $q \ge ns$, $-1 \le t < s \le 1$.

PROOF. Let $p(z) = A_{b,n} + \xi + i\eta = Re^{i\phi}$, then $-\pi/2 < \phi < \pi/2$. Denoting the left hand side of (2.4) by $U_{b,n}(\xi,\eta)$, we get

$$U_{b,n}(\xi,\eta) = q(A_{b,n}+\xi) + nt(A_{b,n}+\xi)\bar{R}^2 + \frac{1 - s^2r^{2n}}{r^{n-1}(1-r^2)}\,[((A_{b,n}+\xi)-A_{1,n})^2 + \eta^2 - D^2_{1,n}]\bar{R}^1, \tag{2.7}$$

and

$$\frac{\partial U_{b,n}}{\partial \eta} = \eta \bar{R}^4\, V_{b,n}(\xi,\eta) \tag{2.8}$$

where

$$V_{b,n}(\xi,\eta) = -2nt(A_{b,n}+\xi) + (D^2_{1,n}+2A_{1,n}(A_{b,n}+\xi)-A^2_{1,n})\left(\frac{1 - s^2r^{2n}}{r^{n-1}(1-r^2)}\right)R + \left(\frac{1 - s^2r^{2n}}{r^{n-1}(1-r^2)}\right)R^3$$

$$= -2ntR\cos\phi + (D^2_{1,n} - A^2_{1,n} + 2A_{1,n}R\cos\phi)\left(\frac{1 - s^2r^{2n}}{r^{n-1}(1-r^2)}\right)R + \left(\frac{1 - s^2r^{2n}}{r^{n-1}(1-r^2)}\right)R^3$$

$$\equiv M_{b,n}(R,\phi)\ (\text{say}).$$

Since, for fixed r with $0 \le r < 1$, $A_{b,n} - D_{b,n}$ decreases as b increases over the interval $[0,1]$, it follows that $R \ge R\cos\phi \ge A_{b,n} - D_{b,n} \ge A_{1,n} - D_{1,n}$. Thus, for all b, where $0 \le b \le 1$,

$$M_{b,n}(R,\phi) \ge R\cos\phi\,[-2nt + (D_{1,n}^2 - A_{1,n}^2 + 2A_{1,n}R\cos\phi + R^2)(\frac{1 - s^2r^{2n}}{r^{n-1}(1 - r^2)})]$$

$$\ge 2R\cos\phi\,[(\frac{1 - s^2r^{2n}}{r^{n-1}(1 - r^2)})(A_{1,n} - D_{1,n})^2 - nt] > 0,$$

for all s,t satisfying $-1 \le t < s \le 1$. Thus $V_{b,n}(\xi,\eta)$ is positive for all points in the disc $\Delta(z)$. Now, (2.6) gives that, for every fixed $\xi$, $U_{b,n}(\xi,\eta)$ is an increasing function of $\eta$ for positive $\eta$ and is a decreasing function of $\eta$ for negative $\eta$. Thus, the minimum of $U_{b,n}(\xi,\eta)$ inside the disc $\Delta(z)$ is attained on the diameter forming part of the real axis. Setting $\eta = 0$ in (2.7), we obtain

$$\min_{-1\le\eta\le1} U_{b,n}(\xi,\eta) = N_{b,n}(R) = (q + \frac{1 - s^2r^{2n}}{r^{n-1}(1 - r^2)})R + \frac{1 + ntr^{n-1}(1 - r^2) - t^2r^{2n}}{r^{n-1}(1 - r^2)}\bar{R}^{1} - 2A_{1,n}\frac{1 - s^2r^{2n}}{r^{n-1}(1 - r^2)}, \quad (2.9)$$

where $R = A_{b,n} + \xi \in [A_{b,n} - D_{b,n}, A_{b,n} + D_{b,n}]$. Thus, the absolute minimum of $N_{b,n}(R)$ in $(0,\infty)$ is attained at

$$R_n^* = \left[\frac{1 + ntr^{n-1}(1 - r^2) - t^2r^{2n}}{1 + qr^{n-1}(1 - r^2) - s^2r^{2n}}\right],$$

and the value of this minimum is equal to

$$N_{b,n}(R_n^*) = \frac{2}{r^{n-1}(1-r^2)}[\sqrt{(1+qr^{n-1}(1-r^2)-s^2r^{2n})(1+ntr^{n-1}(1-r^2)-t^2r^{2n})} - (1-str^{2n})]. \quad (2.10)$$

It is easily seen that $R_n^* < A_{b,n} + D_{b,n}$ but $R_n^*$ may not be always greater than $A_{b,n} - D_{b,n}$. In case $R_n^* \notin [A_{b,n} - D_{b,n}, A_{b,n} + D_{b,n}]$, it can be verified that $N_{b,n}(R)$ increases with R in $[A_{b,n} - D_{b,n}, A_{b,n} + D_{b,n}]$. Thus the minimum of $N_{b,n}(R)$ on the segment $[A_{b,n} - D_{b,n}, A_{b,n} + D_{b,n}]$ is attained at $R_{b,n} = A_{b,n} - D_{b,n}$. The value of this minimum equals

$$N_{b,n}(R_{b,n}) = N_{b,n}(A_{b,n} - D_{b,n}) = \frac{W}{W^*}$$

where W and W* are given by (2.5) and (2.6). Moreover, $N_{b,n}(R_n^*) = N_{b,n}(R_{b,n})$ for those values of q,n,s and t for which $R_{b,n} = R_n^*$. Hence the lemma.

3. THE CLASS $R_n(a;\alpha,\beta)$.

Let $R_n(\alpha,\beta)$ be the class of functions f of the form $f(z) = z + a_{n+1}z^{n+1} + \dots$. which are analytic and satisfy the inequality $|(f'(z)-1)/\{2\beta(f'(z)-\alpha)-(f'(z)-1)\}| < 1$ for some $\alpha,\beta(0 \le \alpha < 1,\ 0 < \beta \le 1)$ and $z \in \Delta$. It is shown in [6] that for $f \in R_n(\alpha,\beta)$, $|a_{n+1}| \le \frac{2\beta(1-\alpha)}{n+1}$. Define

$$R_n(a;\alpha,\beta) = \{f(z) = z + \frac{2\beta(1-\alpha)a}{n+1} z^{n+1} + \dots : f' \in P_n(a;\alpha,\beta),\ 0 \le a \le 1\}.$$

We determine a sharp estimate for the radii of convexity for functions in $R_n(a;\alpha,\beta)$.

THEOREM 1. Let $f \in R_n(a;\alpha,\beta)$, then f is convex in $|z| < r_o$ where $r_o$ is the smallest positive root of the equation

$$(1+ar)^2 + 2a((\beta+\alpha\beta-1) - \beta(1-\alpha)n)r^n(1+r^2) + (2(2\alpha\beta-1)$$
$$- 2\beta(1-\alpha)n + a^2(4\beta-2-2\beta(1-\alpha)n))r^{n+1} + (2\alpha\beta-1)(2\beta-1)r^{2n}(a+r)^2 = 0 \text{ if } R_{a,n} \ge R_n^*$$

and $r_o$ is the smallest positive root of the equation

$$2(1-r^2) + (4(1-\beta-\alpha\beta)n - 2\beta(1-\alpha)(1-n^2))r^{n+1} - (\beta(1-\alpha)(1+n^2)$$
$$+ 2(1-\beta-\alpha\beta)n)r^{n-1}(1+r^4) + 2(2\beta-1)(1-2\alpha\beta)r^{2n}(1-r^2) = 0 \text{ if } R_{a,n} \le R_n^*$$

where

$$R_{a,n} = \frac{1 + ar + (2\alpha\beta-1)ar^n + (2\alpha\beta-1)r^{n+1}}{1 + ar + (2\beta-1)ar^n + (2\beta-1)r^{n+1}},$$

$$R_n^* = \left[\frac{1 + (2\alpha\beta-1)nr^{n-1}(1-r^2) - (2\alpha\beta-1)^2r^{2n}}{1 + (2\beta-1)nr^{n-1}(1-r^2) - (2\beta-1)^2r^{2n}}\right]^{1/2}$$

and $r = |z| < 1$. The bounds are sharp for all admissible values of $\alpha,\beta$,a and n.

PROOF. Since $f \in R_n(a;\alpha,\beta)$, an application of Schwarz's lemma gives

$$f'(z) = \frac{1 + (2\alpha\beta-1)z^{n-1}w(z)}{1 + (2\beta-1)z^{n-1}w(z)}, \tag{3.1}$$

where $w \in B$. Logarithmic differentiation of (3.1) gives

$$1 + z\frac{f''(z)}{f'(z)} = 1 - 2(1-\alpha)\left[\frac{z^n w'(z) + (n-1)z^{n-1}w(z)}{(1+(2\beta-1)z^{n-1}w(z))(1+(2\alpha\beta-1)z^{n-1}w(z))}\right]. \tag{3.2}$$

Applying (2.2) with $s = 2\beta - 1$ and $t = 2\alpha\beta - 1$ to (3.2), we obtain

$$\text{Re}\,\{1 + z\frac{f''(z)}{f'(z)}\} \ge \frac{1}{2\beta(1-\alpha)}\,[\text{Re}((2\beta-1)np(z) + \frac{(2\alpha\beta-1)n}{p(z)})$$
$$- \frac{r^{2n}|(2\beta-1)p(z) - (2\alpha\beta-1)|^2 - |1-p(z)|^2}{r^{n-1}(1-r^2)|p(z)|} + (1 + \frac{(1-\beta-\alpha\beta)n}{\beta(1-\alpha)})], \tag{3.3}$$

where $p(z) = (1 + (2\alpha\beta - 1)z^{n-1}w(z))/(1 + (2\beta - 1)z^{n-1}w(z)$. An application of Lemma 4 with $q = (2\beta - 1)n$, $s = 2\beta - 1$ and $t = 2\alpha\beta - 1$ to (3.3) gives

$$\mathrm{Re}\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geq \begin{bmatrix} \dfrac{1}{\beta(1-\alpha)r^{n-1}(1-r^2)}[\sqrt{M} - (1 - (2\beta-1)(2\alpha\beta-1)r^{2n}) \\ \quad + ((1-\beta)n + \beta(1-\alpha-\alpha n))r^{n-1}(1-r^2)] \quad \text{if } R_{a,n} \leq R_n^* \\ [(1+ar)^2 + 2a((\beta+\alpha\beta-1) - \beta(1-\alpha)n)r^n(1+r^2) + 2(2\alpha\beta - 1 - \beta(1-\alpha)n \\ \quad + a^2(2\beta-1-\beta(1-\alpha)n))r^{n+1} + (2\beta-1)(2\alpha\beta-1)r^{2n}(a+r)^2] \div A, \quad \text{if } R_{a,n} \geq R_n^* \end{bmatrix}, \quad (3.4)$$

where

$$A = (1+ar+(2\beta-1)ar^n+(2\beta-1)r^{n+1})\ (1+ar+(2\alpha\beta-1)ar^n+(2\alpha\beta-1)r^{n+1}$$

and

$$M = (1+(2\alpha\beta-1)nr^{n-1}(1-r^2) - (2\alpha\beta-1)^2r^{2n})(1+(2\beta-1)nr^{n-1}(1-r^2)-(2\beta-1)^2r^{2n}), \quad (3.5)$$

$$R_{a,n} = \frac{1 + ar + (2\alpha\beta - 1)ar^n + (2\alpha\beta - 1)r^{n+1}}{1 + ar + (2\beta - 1)ar^n + (2\beta - 1)r^{n+1}} \text{ and}$$

$$R_n^* = \left[\frac{1+(2\alpha\beta - 1)nr^{n-1}(1 - r^2) - (2\alpha\beta - 1)^2r^{2n}}{1 + (2\beta - 1)nr^{n-1}(1 - r^2) - (2\beta - 1)^2r^{2n}}\right]^{\frac{1}{2}}.$$

Now the theorem follows easily from (3.4).

The functions given by

$$f'(z) = \frac{1 + az + (2\alpha\beta - 1)az^n + (2\alpha\beta - 1)z^{n+1}}{1 + az + (2\beta - 1)az^n + (2\beta - 1)z^{n+1}} \quad \text{if } R_{a,n} \geq R_n^*$$

and

$$f'(z) = \frac{1 + cz + c(2\alpha\beta - 1)z^n + (2\alpha\beta - 1)z^{n+1}}{1 + cz + c(2\beta - 1)z^n + (2\beta - 1)z^{n+1}} \quad \text{if } R_{a,n} \leq R_n^*$$

where c is determined by the relation

$$\frac{1 + cr + c(2\alpha\beta-1)r^n + (2\alpha\beta-1)r^{n+1}}{1 + cr + c(2\beta-1)r^n + (2\beta-1)r^{n+1}} = R_n^* = \left[\frac{1 + (2\alpha\beta-1)nr^{n-1}(1-r^2) - (2\alpha\beta-1)^2r^{2n}}{1 + (2\beta-1)nr^{n-1}(1-r^2) - (2\beta-1)^2r^{2n}}\right]^{1/2}$$

show that the results obtained in the theorem are sharp.

Putting $\beta = 1$ in Theorem 1, we get the following result

COROLLARY 1. Let $f \in R_n(a;\alpha,1) = R_n(a;\alpha)$, then f is convex in $|z| < r_o$ where $r_o$ is the smallest positive root of the equation

$$(1+ar)^2+2a(\alpha-n+n\alpha)r^n(1+r^2)+2(2\alpha-1-n+n\alpha+a^2(1-n+n\alpha))r^{n+1}+(2\alpha-1)r^{2n}(a+r)^2 = 0 \quad \text{if } R_{a,n} \geq R_n^*$$

and $r_o$ is the smallest positive root of the equation

$$2(1 - r^2) + (-4\alpha n - 2(1 - \alpha)(1 - n^2))r^{n+1} - ((1 - \alpha)(1 + n^2) - 2\alpha n)r^{n-1}(1 + r^4) + 2(1 - 2\alpha)r^{2n}(1 - r^2) = 0 \quad \text{if } R_{a,n} \le R_n^*$$

where

$$R_{a,n} = \frac{1 + ar + (2\alpha - 1)ar^n + (2\alpha - 1)r^{n+1}}{1 + ar + ar^n + r^{n+1}},$$

$$R_n^* = \left[\frac{1 + (2\alpha - 1)nr^{n-1}(1 - r^2) - (2\alpha - 1)^2 r^{2n}}{1 + nr^{n-1}(1 - r^2) - r^{2n}}\right]^{1/2}$$

and $|z| = r < 1$. The bounds are sharp for all admissible values of $\alpha$,a and n.

COROLLARY 2. Let $f \in R_n(a;1-\alpha, 1/2) = R_n'(a;\alpha)$, then f is convex in $|z| < r_o$ where $r_o$ is the smallest positive root of the equation

$$(1 + ar)^2 - (n + 1)a\alpha r^n(1 + r^2) - \alpha(2 + n + na^2)r^{n+1} = 0 \quad \text{if } R_{a,n} \ge R_n^*$$

and $r_o$ is the smallest positive root of the equation

$$4(1 - r^2) + 2\alpha(n^2 + 2n - 1)r^{n+1} - \alpha(1 + n)^2 r^{n-1}(1 + r^4) = 0 \text{ if } R_{a,n} \le R_n^*$$

where

$$R_{a,n} = \frac{1 + ar - \alpha ar^n - \alpha r^{n+1}}{1 + ar}, \quad R_n^* = \left[1 - \alpha nr^{n-1}(1 - r^2) - \alpha^2 r^{2n}\right]^{1/2}$$

and $r = |z| < 1$. The bounds are sharp for all admissible values of $\alpha$, a, and n.

The above result is obtained by replacing $\alpha$ by $1 - \alpha$ and $\beta$ by 1/2 in Theorem 1.

It may be noted that, for n = 1, our Corollary 1 gives the corresponding result due to McCarty [2] while Corollary 2 gives the result which was obtained by Goel [7] under the additional restriction $\frac{1}{2} \le \alpha \le 1$.

REMARK 1. Replacing $(\alpha,\beta)$ by (0,1) or by $(0,1-\delta)$ with $0 \le \delta < 1$ or by $(0,(2\delta-1)/2\delta)$ with $\frac{1}{2} < \delta \le 1$, or by $((1-\gamma)/(1+\gamma), (1+\gamma)/2)$ with $0 < \gamma \le 1$, or by $((1-\delta+2\gamma\delta)/(1+\delta), (1+\delta)/2)$ with $0 \le \gamma < 1$ and $0 < \delta \le 1$, we get the estimates for the radii of convexity for functions of the form $f(z) = z + a_{n+1}z^{n+1} + \dots$ with fixed (n + 1)th coefficient of the classes introduced and studied by MacGregor [8], Shaffer [9], Goel [10], Caplinger and Causey [11] and the authors [12] respectively.

## 4. THE CLASS $S_n^*(a;\alpha,\beta)$.

Let $S_n^*(\alpha,\beta)$ be the class of functions g of the form $g(z) = z + a_{n+1}a^{n+1} + \dots$ which are analytic and satisfy the inequality $|(zg'(z)/g(z)-1)/\{2\beta(zg'(z)/g(z)-\alpha)-(zg'(z)/g(z)-1)\}| < 1$, for some $\alpha,\beta(0 \le \alpha < 1, 0 < \beta \le 1)$ and $z \in \Delta$. The authors [13] have shown that for $g \in S_n^*(\alpha,\beta)$, $|a_{n+1}| \le \frac{2\beta(1 - \alpha)}{n}$.

Define

$$S_n^*(a;\alpha,\beta) = \{g(z) = z + \frac{2\beta(1-\alpha)a}{n} z^{n+1} + \dots : zg'/g \in P_n(a;\alpha,\beta), \quad 0 \le a \le 1\}.$$

Now, we determine a sharp estimate for the radii of convexity for the functions in $S_n^*(a;\alpha,\beta)$.

THEOREM 2. Let $g \in S_n^*(a;\alpha,\beta)$, then g is convex in $|z| < r_o$ where $r_o$ is the smallest positive root of the equation

$$(1 + ar)^2 + 2a(2\alpha\beta - 1 - \beta(1-\alpha)n)r^n(1 + r^2) + 2((2\alpha\beta - 1)(1 + a^2) + \beta(1-\alpha)(a^2(1-n) - n - 1))r^{n+1} + (2\alpha\beta - 1)^2 r^{2n}(a + r)^2 = 0 \text{ if } R_{a,n} \ge R_n^*$$

and $r_o$ is the smallest positive root of the equation

$$2(1 - 2\alpha\beta)^2 r^{2n}(1 - r^2) + n(\beta(1-\alpha)n + 2 - 4\alpha\beta)r^{n-1}(1 + r^4) + 2(\beta(1-\alpha)(2 - n^2) - 2n(1 - 2\alpha\beta))r^{n+1} + 2r^2 - 2 = 0 \quad \text{if } R_{a,n} \le R_n^*$$

where

$$R_{a,n} = \frac{1 + ar + (2\alpha\beta - 1)ar^n + (2\alpha\beta - 1)r^{n+1}}{1 + ar + (2\beta - 1)ar^n + (2\beta - 1)r^{n+1}},$$

$$R_n^* = \left[\frac{1 + (2\alpha\beta - 1)nr^{n-1}(1 - r^2) - (2\alpha\beta - 1)^2 r^{2n}}{1 + (2\beta(1-\alpha) + (2\beta - 1)n)r^{n-1}(1 - r^2) - (2\beta - 1)^2 r^{2n}}\right]^{1/2}$$

and $r = |z| < 1$. The bounds are sharp for all admissible values of $\alpha,\beta$,a and n.

PROOF. Since $g \in S_n^*(a;\alpha,\beta)$, an application of Schwarz lemma gives

$$z\frac{g'(z)}{g(z)} = \frac{1 + (2\alpha\beta - 1)z^{n-1}w(z)}{1 + (2\beta - 1)z^{n-1}w(z)}, \tag{4.1}$$

where $w \in B$ and for all $z \in \Delta$.

Differentiating (4.1) logarithmically, we have

$$1 + z\frac{g''(z)}{g'(z)} = \frac{1 + (2\alpha\beta - 1)z^{n-1}w(z)}{1 + (2\beta - 1)z^{n-1}w(z)} - 2\beta(1-\alpha)\left[\frac{z^n w'(z) + (n-1)z^{n-1}w(z)}{(1 + (2\beta - 1)z^{n-1}w(z))(1 + (2\alpha\beta - 1)z^{n-1}w(z)}\right]. \tag{4.2}$$

Applying (2.2) with $s = 2\beta - 1$ and $t = 2\alpha\beta - 1$ to (4.2), we obtain

$$\text{Re}(1 + z\frac{g''(z)}{g'(z)}) \ge \frac{1}{2\beta(1-\alpha)}[\text{Re}(((2\beta - 1)n + 2\beta(1-\alpha))p(z) + \frac{(2\alpha\beta - 1)n}{p(z)}) - \frac{r^{2n}|(2\beta - 1)p(z) - (2\alpha\beta - 1)|^2 - |1 - p(z)|^2}{r^{n-1}(1 - r^2)|p(z)|} - \frac{(\beta + \alpha\beta - 1)n}{\beta(1-\alpha)}], \tag{4.3}$$

where $p(z) = (1 + (2\alpha\beta - 1)z^{n-1}w(z))/(1 + (2\beta - 1)z^{n-1}w(z))$. Now, an application of

Lemma 4 with $q = 2\beta(1-\alpha) + (2\beta - 1)n$, $s = 2\beta - 1$ and $t = 2\alpha\beta - 1$ to (4.3) gives the required results easily.

The functions given by

$$z\,\frac{g'(z)}{g(z)} = \frac{1 + az + (2\alpha\beta - 1)az^n + (2\alpha\beta - 1)z^{n+1}}{1 + az + (2\beta - 1)az^n + (2\beta - 1)z^{n+1}}$$

and

$$z\,\frac{g'(z)}{g(z)} = \frac{1 + cz + (2\alpha\beta - 1)cz^n + (2\alpha\beta - 1)z^{n+1}}{1 + cz + (2\beta - 1)cz^n + (2\beta - 1)z^{n+1}}$$

where c is determined by the relation

$$\frac{1 + cr + (2\alpha\beta - 1)cr^n + (2\alpha\beta - 1)r^{n+1}}{1 + cr + (2\beta - 1)cr^n + (2\beta - 1)r^{n+1}} = $$

$$= R_n^* = \left[\frac{1 + (2\alpha\beta - 1)nr^{n-1}(1 - r^2) - (2\alpha\beta - 1)^2 r^{n+1}}{1 + (2\beta(1-\alpha) + (2\beta - 1)n)r^{n-1}(1 - r^2) - (2\beta - 1)^2 r^{n+1}}\right]^{1/2}$$

show that the results obtained in the theorem are sharp.

Taking $\beta = 1$ in Theorem 2, we get the following result:

COROLLARY 3. Let $g \in S_n^*(a;\alpha,\beta) \equiv S_n^*(a,\alpha)$, then g is convex in $|z| < r_o$ where $r_o$ is the smallest positive root of the equation

$$(1 + ar)^2 + 2a(2\alpha - 1 - (1 - \alpha)n)r^n(1 + r^2) + 2((2\alpha - 1)(1 + a^2)$$
$$+ (1 - \alpha)(a^2(1 - n) - n - 1))r^{n+1} + (2\alpha - 1)^2 r^{2n}(1 + r)^2 = 0 \quad \text{if } R_{a,n} \ge R_n^*$$

and $r_o$ is the smallest positive root of the equation

$$2(1 - 2\alpha)^2 r^{2n}(1 - r^2) + n((1 - \alpha)n + 2 - 4\alpha)r^{n-1}(1 + r^4) + 2((1 - \alpha)(2 - n^2)$$
$$- 2n(1 - 2\alpha))r^{n+1} + 2r^2 - 2 = 0 \quad \text{if } R_{a,n} \le R_n^*$$

where

$$R_{a,n} = \frac{1 + ar + (2\alpha - 1)ar^n + (2\alpha - 1)r^{n+1}}{1 + ar + ar^n + r^{n+1}},$$

$$R_n^* = \left[\frac{1 + (2\alpha - 1)nr^{n-1}(1 - r^2) - (2\alpha - 1)^2 r^{2n}}{1 + (2 - 2\alpha + n)r^{n-1}(1 - r^2) - r^{2n}}\right]^{1/2},$$

and $|z| = r < 1$. The bounds are sharp for all admissible values of $\alpha$, a and n.

For n = 1, Corollary 3 gives the corresponding result due to McCarty [8] which includes the result obtained by Tepper [1].

COROLLARY 4. Let $g \in S_n^*(a;1 - \alpha,1/2) \equiv S_n^{**}(a,\alpha)$, then g is convex in $|z| < r_o$ where $r_o$ is the smallest positive root of the equation

$$(1+ar)^2 - \alpha(n+2)ar^n(1+r^2) - 2\alpha(2+(1+n)(1+a^2))r^{n+1} + \alpha^2 r^{2n}(a+r)^2 = 0 \quad \text{if } R_{a,n} \ge R_n^*$$

and $r_o$ is the smallest positive root of the equation

$$4\alpha^2 r^{2n}(1-r^2) + n\alpha(n+4)r^{n-1}(1+r^4) + 2\alpha(2-4n-n^2)r^{n+1} + 4r^2 - 4 = 0 \quad \text{if } R_{a,n} \le R_n^*$$

where

$$R_{a,n} = \frac{1 + ar - a\alpha r^n - \alpha r^{n+1}}{1 + ar} \quad \text{and} \quad R_n^* = \left[\frac{1 - n\alpha r^{n-1}(1 - r^2) - \alpha^2 r^{2n}}{1 + \alpha r^{n-1}(1 - r^2)}\right]^{1/2}$$

and $|z| = r < 1$. The bounds are sharp for all admissible values of $\alpha$, a and n.

The above result is obtained by replacing $\alpha$ by $1 - \alpha$ and $\beta$ by 1/2 in Theorem 2.

REMARK 2. Replacing $(\alpha,\beta)$ by $(0,1/2)$, or by $(0,(2\delta-1)/2\delta)$ with $\frac{1}{2} < \delta \le 1$, or by $((1-\gamma)/(1+\gamma), (1+\gamma)/2)$ with $0 < \gamma \le 1$, we obtain the estimates for the radii of convexity for functions of the form $f(z) = z + a_{n+1}z^{n+1} + \ldots$ with fixed (n+1)th coefficient of the classes introduced and studied by Singh [14,15] and Padmanabhan [16] respectively.

REMARK 3. The result obtained by the authors [11] regarding sharp radii of convexity estimates for starlike functions with (n+1)th fixed coefficient can be obtained from Theorem 2 by putting $(\alpha,\beta) = (0,1)$.

REMARK 4. Setting n = 1 in Theorem 1 and Theorem 2, we get the corresponding sharp estimates for the radii of convexity for functions in $R_1(a;\alpha,\beta) \equiv R_a(\alpha,\beta)$, $S_1^*(a;\alpha,\beta) \equiv S_a^*(\alpha,\beta)$ determined in [3].

REMARK 5. The sharp estimates for the radii of convexity for functions in the classes $R_n(\alpha,\beta)$ and $S_n^*(\alpha,\beta)$ can respectively be determined from Theorem 1 and Theorem 2 by fixing a = 1, which in turn give the results obtained in [6] and [17] for n = 1.

REMARK 6. By setting a = 0 in Theorems 1 and 2, we may obtain sharp radius of convexity results for functions with symmetric gaps in the classes $R_n(\alpha,\beta)$ and $S_n^*(\alpha,\beta)$.

## REFERENCES


1. TEPPER, T.P. On the radius of convexity and boundary distortion of schlicht functions, Trans. Amer. Math. Soc. 150 (1970), 519-528.

2. McCARTY, C.P. Two radius of convexity problems, Proc. Amer. Math. Soc. 42 (1974), 153-160.

3. JUNEJA, O.P. and MOGRA, M.L. Radii of convexity for certain classes of univalent analytic functions, Pacific J. Math. (2) 78 (1978), 359-368.

4. MOGRA, M.L. and JUNEJA, O.P. A radius of convexity problem, Bull. Austral. Math. Soc. 24 (1981), 381-388.

5. SINGH, V. and GOEL, R.M. On radii of convexity and starlikeness of some classes of functions, J. Math. Soc. Japan 23 (1971), 323-339.

6. MOGRA, M.L. On a class of univalent functions whose derivatives have a positive real part, Riv. Math. Univ. Parma (to appear).

7. GOEL, R.M. A class of univalent functions with fixed second coefficient, J. Math. Sci. 4 (1969), 85-92.

8. MacGREGOR, T.H. Functions whose derivatives have positive real parts, Trans. Amer. Math. Soc. 104 (1962), 532-537.

9. SHAFFER, D.B. Distortion theorems for a special class of analytic functions, Proc. Amer. Math. Soc. 39 (1973), 281-287.

10. GOEL, R.M. A class of univalent functions whose derivatives have positive real part in the unit disc, Nieuw Arch. Wisk 15 (1967), 55-63.

11. CAPLINGER, T.R. and CAUSEY, W.M. A class of univalent functions, Proc. Amer. Math. Soc. 39 (1973), 357-361.

12. JUNEJA, O.P. and MOGRA, M.L. A class of univalent functions, Bull. Sci. Math. 103 (1979), 435-447.

13. MOGRA, M.L. and JUNEJA, O.P. Coefficient estimatex for starlike functions, Bull. Austral. Math. Soc. (3) 16 (1977), 415-425.

14. SINGH, R. On a class of starlike functions, Compositio Math. 19 (1967), 78-82.

15. SINGH, R. On a class of starlike functions II, Ganita 19 (1968), 103-110.

16. PADMANABHAN, K.S. On certain classes of starlike functions in the unit disc, J. Indian Math. Soc. 32 (1968), 89-103.

17. JUNEJA, O.P. and MOGRA, M.L. On starlike functions of order $\alpha$ and type $\beta$, Rev. Roumaine Math. Pures Appl. 23 (1978), 751-765.