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ON A NON-SELF ADJOINT EXPANSION FORMULA

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ABSTRACT. This paper develops a formula of inversion for an integral transform of the kind similar to that associated with the names of Kontorovich and Lebedev except that the kernel involves the Neumann function $Y_u(kr)$ and the variable r varies over the infinite interval $a \leq r < \infty$ where $a > 0$. The transform is useful in the investigation of functions that satisfy the Helmholtz equation and a condition of radiation at infinity. The formula established is expressed entirely in terms of series expansions and replaces earlier inversion formulas that require the evaluation of contour integrals.

KEY WORDS AND PHRASES. *Integral transforms, Eigenfunction expansions, Bessel functions.*
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1. INTRODUCTION.

In a previous paper Naylor and Chang [1] the author gave a formula of inversion for an integral transform of the type associated with the names of Kontorovich and Lebedev. The transform in question involves the truncated infinite interval $a \leq r < \infty$ where $a > 0$ and was defined by the equation

$$F(u) = \int_a^\infty Y_u(kr)f(r)\frac{dr}{r} . \tag{1.1}$$

Here $k > 0$ and $Y_u(kr)$ denotes the Neumann function, the notation being that of Watson [2]. This transform is useful in the solution of certain boundary problems involving the Helmholtz equation and the radiation condition

$$\lim_{r\to\infty} r^{\frac{1}{2}}[f'(r) - ikf(r)] = 0 . \tag{1.2}$$

The author has considered also the alternative transform $G(u)$ defined by the equation

$$G(u) = \int_a^\infty [J_u(kr)H_u^{(1)}(ka) - J_u(ka)H_u^{(1)}(kr)]f(r)\frac{dr}{r}$$

where $H_u^{(1)}$ denotes the Hankel function of the first kind. This transform can also be applied to boundary problems of the above type but its formula of inversion involves either a contour integral, as obtained in Naylor [3], or a series and an integral, as derived in Naylor [4]. Likewise the inversion formula established in Naylor and Chang [1] for the transform (1.1) also required an integral term as well as a series. In this paper a simpler inversion formula is found for (1.1) which expresses $f(r)$ as the sum of two series, the integral term being absent.

The functions appearing in the series derived in this paper are the Bessel functions $Y_{u_n}(kr)$ where $u_1, u_2, \dots$ denote the zeros of the function $Y_u(kr)$ regarded as a function of the order u. These functions are not referred to as eigenfunctions since they do not form an orthogonal set on the interval (a,∞). The zeros of $Y_u(ka)$ for given positive ka were discussed in Naylor [5] where it was shown that there exist three infinite sets of zeros as follows:

(i) an infinite set of real negative zeros u_n which for large n are given by the asymptotic formula

$$u_n \sim -(n+\tfrac{1}{2}) + [ka/(2n+1)]^{2n+1} \tag{1.3}$$

(ii) two infinite sets of complex zeros $u_n' = R_n e^{\pm i\theta_n}$ located in the first and fourth quadrants of the complex u-plane and given for large n by the asymptotic formulas

$$\theta_n \sim \frac{\pi}{2}\left[1 - \frac{1}{2\log(2R_n/kae)}\right] \tag{1.4}$$

$$R_n \log(2R_n/kae) \sim (n-\tfrac{1}{4})\pi . \tag{1.5}$$

The zeros of all three sets are simple and there are no purely imaginary zeros.

The actual expansion constructed in this paper is stated in the following theorem:

THEOREM. Suppose that $f(r)$ is continuous for $r \geq a > 0$ and that $r^{-1}f(r) \in L(a,\infty)$. Let the transform $F(u)$ be defined for positive values of k by means of equation (1.1). Then, if $r > a$,

$$f(r) = \pi \sum_{u_n} \frac{uJ_u(ka)J_u(kr)\cot u\pi\, F(u)}{(\partial/\partial u)Y_u(ka)} + \pi \lim_{c\to 0} \sum_{u_n'} \frac{uJ_u(ka)J_u(kr)\cot u\pi\, F(u)e^{cu^2}}{(\partial/\partial u)Y_u(ka)} \tag{1.6}$$

where the summations together extend over all of the zeros of $Y_u(ka)$ regarded as a function of u. The first series in (1.6) includes all of the real zeros u_n and the second series includes all of the complex zeros u_n'. The exponential function appearing in the second series in (1.6) is a summability factor, the parameter c tending to zero through positive values. Because of the non-self adjoint character of the underlying expansion problem it is not possible to deduce the above expansion from Titchmarsh's treatise [6] on this subject, neither does it appear possible to obtain such a formula without introducing a summability factor of one kind or another.

2. THE REAL RESIDUE SERIES.

In order to obtain the formula quoted in the above theorem it will be necessary to appeal to the following formula which was established in Naylor and Chang [1], equation (45),

$$f(r) = \frac{1}{2i}\lim_{c\to 0}\int_L \frac{[Y_u(kr)J_u(ka) - Y_u(ka)J_u(kr)]F(u)e^{cu^2}\,du}{Y_u(ka)}$$

$$+ \pi\lim_{c\to 0}\sum_{u_n'} \frac{uY_u(kr)J_u(ka)F(u)e^{cu^2}}{(\partial/\partial u)Y_u(ka)} \quad . \tag{2.1}$$

The summation in (2.1) extends over all of the complex zeros u_n' , all of which are located in the half plane $\mathrm{Re}(u) > 0$.

It should be noted that the series appearing in (2.1) is different from either of those that are present in the formula (1.6) to be proved.

The method to be adopted requires the integral in (2.1) in which L denotes the imaginary axis of the complex u-plane, to be evaluated by means of the calculus of residues. The asymptotic behaviour of the integrand prevents a direct evaluation of the integral in its present form, however the integral can be evaluated if the cross product of Bessel functions appearing in the numerator of the integrand is expressed in a different form. The resulting integral can then be expressed as the difference of two integrals each of which can be evaluated in terms of the residues at the poles, one in terms of the residues at the poles in the half plane $\mathrm{Re}(u) > 0$, the other in terms of the residues at the poles in the half plane $\mathrm{Re}(u) < 0$.

With this aim in view we appeal to the formula, Magnus et al [7] p. 66,

$$Y_u(x) = J_u(x)\cot u\pi - J_{-u}(x)\operatorname{cosec} u\pi \tag{2.2}$$

from which it is seen that

$$Y_u(kr)J_u(ka) - Y_u(ka)J_u(kr) = -[J_{-u}(kr)J_u(ka) - J_{-u}(ka)J_u(kr)]\operatorname{cosec} u\pi$$

so that

$$I_1 - I_2 = \frac{1}{2i}\lim_{c\to 0}\int_L \frac{[Y_u(kr)J_u(ka) - Y_u(ka)J_u(kr)]F(u)e^{cu^2}\,du}{Y_u(ka)} \tag{2.3}$$

where

$$I_1 = \frac{1}{2i}\lim_{c\to 0}\int_L \frac{J_{-u}(ka)J_u(kr)F(u)e^{cu^2}\,du}{Y_u(ka)\sin u\pi} \tag{2.4}$$

$$I_2 = \frac{1}{2i}\lim_{c\to 0}\int_L \frac{J_{-u}(kr)J_u(ka)F(u)e^{cu^2}\,du}{Y_u(ka)\sin u\pi} \quad . \tag{2.5}$$

The quantity I_1 will be determined in terms of the residues at the poles located in the left half plane $\mathrm{Re}(u) < 0$. There are two such series of poles corresponding to the real negative zeros of $Y_u(ka)$ and to the negative zeros of $\sin u\pi$. If we write $u = Re^{i\theta}$ we see that the summability factor $\exp(cu^2)$ has modulus $\exp(cR^2\cos 2\theta)$ which tends to zero as $R \to \infty$ in the sectors $\frac{\pi}{2} \le |\theta| < \frac{3\pi}{4}$ but which diverges as $R \to \infty$ in the sectors $\frac{3\pi}{4} < |\theta| \le \pi$. To evaluate I_1 the path L is first deformed onto the path W which consists of the rays $\theta = \pm\frac{3\pi}{4}$, as depicted in Figure 1. This process leads to the equation

$$I_1 = \frac{1}{2i}\lim_{c\to 0}\int_W \frac{uJ_{-u}(ka)J_u(kr)F(u)e^{cu^2}\,du}{Y_u(ka)\sin u\pi} \tag{2.6}$$

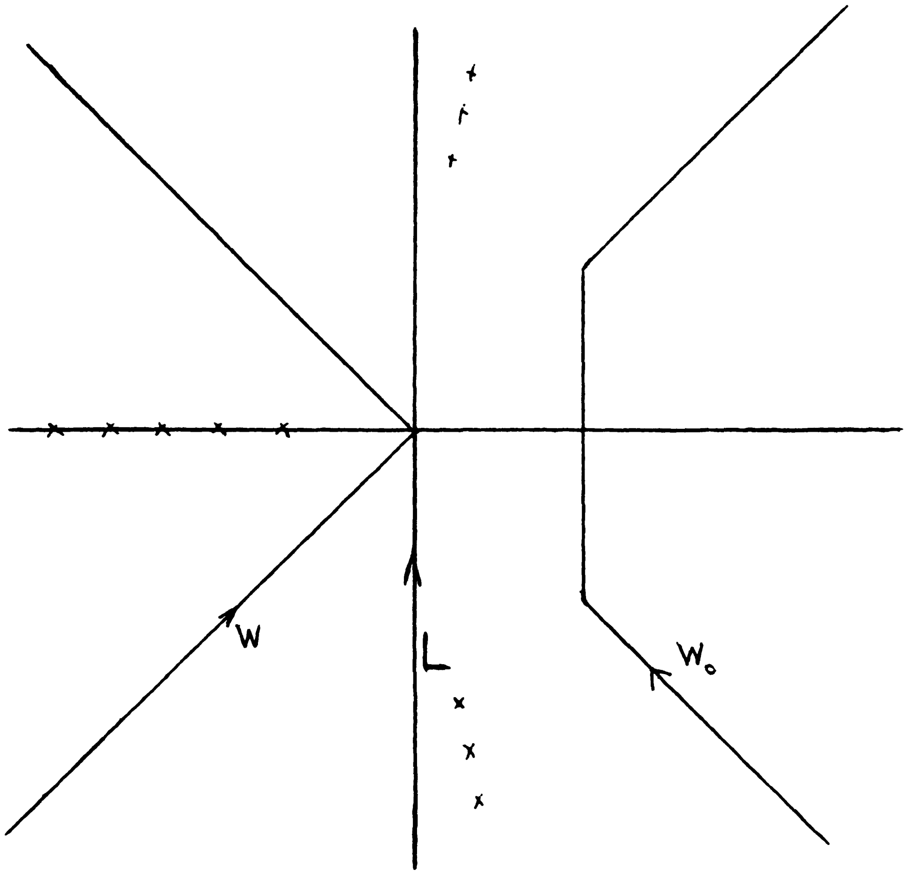

Figure 1

The above procedure will be justified by an inspection of the asymptotic behaviour of the integrand as $u \to \infty$ in the region between L and W. This is carried out in the following paragraphs where it is also proved that the integral (2.6) is uniformly convergent for all $c \geq 0$. We may then set $c = 0$ in the integral which can then be evaluated by closing the contour on the left hand side of W and taking the residues at the poles.

To determine the asymptotic behaviour of the Bessel functions appearing in (2.6) we employ the formula

$$J_u(x) = \frac{(x/2)^u}{\Gamma(u+1)} [1 + O(u^{-1})] \ . \qquad (2.7)$$

This formula holds for fixed x and u large and bounded away from the negative integers. Upon using this result together with the identity $\Gamma(u)\Gamma(1-u) = \pi \operatorname{cosec} u\pi$ we find that

$$uJ_{-u}(ka)J_u(kr)\operatorname{cosec} u\pi = \frac{1}{\pi} (r/a)^u[1 + O(u^{-1})] \ . \qquad (2.8)$$

A bound on the function $F(u)$ present in (2.6) may be obtained from (1.1) by applying the Schwarz inequality which leads to the inequality

$$|F(u)| \leq ||f(r)|| \ ||Y_u(kr)|| \qquad (2.9)$$

where

$$||f(r)|| = (\int_a^\infty |f(r)|^2 \ r^{-1}dr)^{\frac{1}{2}} \ . \qquad (2.10)$$

The quantity $||Y_u(kr)||$ can be obtained from Naylor and Chang [1], equation (34)

$$\pi R^2 \sin 2\theta \int_a^\infty r^{-1}|Y_u(kr)|^2 dr = \sinh(\pi R \sin\theta) + ka \operatorname{Im} Y_u(ka) Y'_{\bar{u}}(ka) \tag{2.11}$$

where $u = Re^{i\theta}$, $\bar{u} = Re^{-i\theta}$.

The behaviour of $Y_u(x)$ can be obtained from (2.2) in conjunction with (2.7) and the identity $\Gamma(u)\Gamma(1-u) = \pi \operatorname{cosec} u\pi$. This yields the equation

$$Y_u(x) = \left[\frac{(x/2)^u \cot u\pi}{\Gamma(u+1)} - \frac{1}{\pi}(x/2)^{-u}\Gamma(u)\right][1 + O(u^{-1})] \tag{2.12}$$

for large u bounded away from the integers. The Γ-functions occurring in the above formulas may be estimated for large values of u by means of Stirling's formula, Magnus et al [7], p. 12,

$$\Gamma(u) = (2\pi/u)^{\frac{1}{2}} \exp(u \log u - u)[1 + O(u^{-1})] \tag{2.13}$$

which holds as $u \to \infty$ in $|\arg u| \le \pi - \delta$. It follows from this formula that

$$|\Gamma(u)| = (2\pi/R)^{\frac{1}{2}} \exp[R\cos\theta \log(R/e) - R\theta \sin\theta][1 + O(R^{-1})] \tag{2.14}$$

so that $\Gamma(u) \to 0$ as $R \to \infty$ in the sectors $\frac{\pi}{2} < |\theta| < \pi$ since $\cos\theta < 0$ whilst $\theta \sin\theta \ge 0$ there. It follows from (2.14) that (2.12) simplifies in these sectors to give the equation

$$Y_u(x) = \frac{(x/2)^u \cot u\pi}{\Gamma(u+1)}[1 + O(u^{-1})] \tag{2.15}$$

for large u and $\frac{\pi}{2} \le |\arg u| \le \pi - \delta$. The form of the derivative $Y'_u(x)$ for large u can be found from the identity

$$2Y'_u(x) = Y_{u-1}(x) - Y_{u+1}(x). \tag{2.16}$$

Upon substituting the values of Y_{u-1} and Y_{u+1} obtained from equation (2.15) we find, after slight reduction, that

$$2Y'_u(x) = \frac{u(x/2)^{u-1} \cot u\pi}{\Gamma(u+1)}[1 + O(u^{-1})]$$

so that, by virtue of (2.15),

$$xY'_u(x) = uY_u(x)[1 + O(u^{-1})]. \tag{2.17}$$

After inserting this expression, with x replaced by ka therein, into equation (2.11), we find, since $\operatorname{Im} u = R\sin\theta$, that

$$\int_a^\infty r^{-1}|Y_u(kr)|^2 dr = \frac{\sinh(R\sin\theta)}{2\pi R^2 \sin\theta \cos\theta} - \frac{|Y_u(ka)|^2}{2\pi R\cos\theta} < \frac{|Y_u(ka)|^2}{|2\pi R\cos\theta|}. \tag{2.18}$$

This applies in the sectors $\frac{\pi}{2} < |\theta| \le \pi - \delta$ since $\cos\theta < 0$ there. It follows from (2.9) that

$$\frac{F(u)}{Y_u(ka)} = O(|R\cos\theta|^{-\frac{1}{2}}) \tag{2.19}$$

for $\frac{\pi}{2} < |\theta| \le \pi - \delta$. The bound (2.19) breaks down in the vicinity of the imaginary axis where, however the function $F(u)$ may be estimated with the aid of the inequality

$$|Y_u(kr)| \le 2(\pi kr)^{-\frac{1}{2}} \cosh(\tfrac{\pi}{2}R\sin\theta). \tag{2.20}$$

This result, which holds for $|\mathrm{Re}(u)| \le \frac{1}{3}$ was established in appendix 1 of Naylor and Chang [1]. On using this bound in (1.1) we find, since $r^{-3/2} f(r) \in L(a,\infty)$, that

$$F(u) = O(\exp|\tfrac{\pi}{2} R \sin \theta|) \tag{2.21}$$

for $|\mathrm{Re}(u)| \le \frac{1}{3}$.

The value of $Y_u(ka)$ for large u in the strip $-\frac{1}{3} \le \mathrm{Re}(u) \le 0$ follows from (2.15) and (2.14). Since $|\cot u\pi| \to 1$ as $u \to \infty$ in this strip we find the equation

$$|Y_u(ka)| \sim (2\pi R)^{-\frac{1}{2}} \exp[-R\cos\theta \log(2R/kae) + R\theta\sin\theta]$$

$$\gtrsim (2\pi R)^{-\frac{1}{2}} \exp(R\theta\sin\theta). \tag{2.22}$$

On combining this bound with (2.17) we find that

$$\frac{F(u)}{Y_u(ka)} = O\{R^{\frac{1}{2}} \exp[(\tfrac{\pi}{2} - |\theta|)|R\sin\theta|]\}$$

as $u \to \infty$ in the strip $-\frac{1}{3} \le \mathrm{Re}(u) \le 0$. Since $|\cos\theta| \le 1/3R$ then $\frac{\pi}{2} - |\theta|$ is $O(R^{-1})$ and the above simplifies to yield the equation

$$F(u)/Y_u(ka) = O(R^{\frac{1}{2}}) \tag{2.23}$$

as $u \to \infty$ in the strip $-\frac{1}{3} \le \mathrm{Re}(u) \le 0$.

An inspection of (2.8) and (2.19) or (2.23), as the case may be, reveals that the integrand in (2.4) is

$$O\{R^{-\frac{1}{2}} \exp[R\cos\theta \log(r/a) + cR^2\cos 2\theta]\}$$

in the strip $-\frac{1}{3} \le \mathrm{Re}(u) \le 0$ and,

$$O\{|R\cos\theta|^{-\frac{1}{2}} \exp[R\cos\theta \log(r/a) + cR^2\cos 2\theta]\}$$

elsewhere in the sectors $\frac{\pi}{2} \le |\theta| \le \frac{3\pi}{4}$. Both these bounds tend to zero sufficiently rapidly as $R \to \infty$ to permit the path L in (2.4) to be deformed onto the path W and this establishes the validity of (2.6).

On the path W, $\theta = \pm\frac{3\pi}{4}$ and the modulus of the summability factor is equal to unity. The integrand in (2.6) is then

$$O\{R^{-\frac{1}{2}} \exp[-\frac{R}{\sqrt{2}} \log(r/a)]\}$$

so that I_1 is absolutely and uniformly convergent for all $c \ge 0$ and the limiting value may be obtained by setting $c = 0$ in the integrand. This leads to the equation

$$I_1 = \frac{1}{2i}\int_W \frac{uJ_{-u}(ka)J_u(kr)F(u)du}{Y_u(ka)\sin u\pi}. \tag{2.24}$$

The next step is to evaluate the above integral in terms of the residues at the poles situated to the left of W. All of these poles are located on the negative real axis. The relevant zeros of $Y_u(ka)$ are located at the points u_n which for large n are asymptotic to the values $-(n+\frac{1}{2})$ whilst those of $\sin u\pi$ occur at the points $u = -n$ where, in each case, $n = 1, 2, \ldots$ On closing the contour in (2.24) on the left hand side of W and evaluating the residues at the stated poles we find the formula

$$I_1 = \pi \sum_{u_n} \frac{uJ_{-u}(ka)J_u(kr)F(u)}{\sin u\pi\,(\partial/\partial u)Y_u(ka)} - \sum_{n=1}^{\infty} \frac{nJ_n(ka)J_n(kr)F(n)}{Y_n(ka)} . \tag{2.25}$$

The first series occurring in the above equation is summed over the real negative zeros u_n of $Y_u(ka)$ and we have used the properties $J_{-n} = (-1)^n J_n$, $Y_{-n} = (-1)^n Y_n$ and $F(-n) = (-1)^n F(n)$ to simplify the second series appearing therein.

To justify the above procedure the contour of integration must be closed on the left side of W by means of a sequence of curves C_n which recede to infinity and which avoid the poles of the integrand. A suitable curve C_n can be made up of the two circular arcs $u = Re^{i\theta}$, $\frac{3\pi}{4} \le |\theta| \le \pi - \delta$ connected by the part of the straight line $u = -(n + \frac{1}{4}) + is$ located inside the wedge $\pi - \delta \le |\theta| \le \pi$. The radius R of the arcs is chosen so that $R\cos\delta = n + \frac{1}{4}$ to ensure that a continuous curve is formed. It is clear that for large n the curves C_n constructed in this way will avoid the poles of the integrand, since the latter are positioned at the points where $u = -n$ and $u \sim -(n + \frac{1}{2})$.

The asymptotic behaviour of the integrand in the sectors $\frac{3\pi}{4} \le |\theta| \le \pi - \delta$ can be obtained from (2.8) and (2.19) both of which apply in these sectors. These equations show that the integrand in I_1 is

$$O\{|R\cos\theta|^{-\frac{1}{2}} \exp[R\cos\theta \log(r/a)]\} \tag{2.26}$$

which tends to zero as $R \to \infty$ since $\cos\theta < 0$ and $\log(r/a) > 0$. It will be verified that the above bound also applies in the wedge $\pi - \delta \le |\theta| \le \pi$ and therefore in the entire sector $\frac{3\pi}{4} \le |\theta| \le \pi$.

To estimate the asymptotic behaviour of the integrand in the wedge $\pi - \delta \le |\theta| \le \pi$ we appeal to the formula (2.12) in which the Γ-functions of argument u must be replaced by ones of argument $-u$ in order to permit the use of Stirling's formula in this region. This can be carried out with the help of the identity $\Gamma(u)\Gamma(1-u) = \pi\,\mathrm{cosec}\,u\pi$ which enables equation (2.12) to be converted to the desired form:

$$Y_u(x) = [-\frac{1}{\pi}(x/2)^u\Gamma(-u)\cos u\pi + \frac{(x/2)^{-u}}{u\Gamma(-u)\sin u\pi}][1 + O(u^{-1})] . \tag{2.27}$$

To apply Stirling's formula (2.13) in the stated sector we ensure that $|\arg(-u)| \le \pi - \delta$ therein by writing $-u = Re^{i\psi}$ where $\psi = \theta - \pi$ for $\pi - \delta \le \theta \le \pi$ and $\psi = \theta + \pi$ for $-\pi \le \theta \le -\pi + \delta$. With this definition of ψ, $|\psi| \le \delta$ and (2.15) gives the formula

$$|\Gamma(-u)| = (2\pi/R)^{\frac{1}{2}} \exp[R\cos\psi \log(R/e) - R\psi\sin\psi][1 + O(R^{-1})] . \tag{2.28}$$

This diverges as $R \to \infty$ in the wedge $\pi - \delta \le |\theta| \le \pi$ since $|\psi| \le \delta$ and $\cos\psi > 0$ there. On the lines $u = -(n+\frac{1}{4}) + is$, $|\sin u\pi| = |\cos u\pi| = (\frac{1}{2}\cosh 2s\pi)^{\frac{1}{2}}$ so on collecting these results it is seen that the first term on the right hand side of (2.27) is the dominant one so that

$$Y_u(x) = -\frac{1}{\pi}(x/2)^u\Gamma(-u)\cos u\pi[1 + O(u^{-1})] .$$

This is the same formula as (2.15), since $\Gamma(u)\Gamma(1-u) = \pi\,\mathrm{cosec}\,u\pi$, so that (2.17), (2.18) and therefore (2.19) also hold in the wedge $\pi - \delta \le |\theta| \le \pi$. Hence the bound (2.26) is also valid throughout the sector $\frac{3\pi}{4} \le |\theta| \le \pi$, as stated above.

Since (2.26) tends to zero sufficiently rapidly as $R \to \infty$ in the sectors $\frac{3\pi}{4} \le |\theta| \le \pi$ the integrals along the sequence on curves C_n used to close the contour tend to zero as their radii tend to infinity and this verifies the validity of (2.25).

3. THE COMPLEX RESIDUE SERIES.

In this section the integral I_2 defined by equation (2.5) will be transformed by closing the contour on the right hand side and taking the residues at the poles of the integrand located in the half plane $\mathrm{Re}(u) > 0$. These poles occur at the complex zeros u_n' of $Y_u(ka)$ and at the points $u = 1,2,3, \ldots$ which arise from the positive zeros of $\sin u\pi$.

We commence with the equation

$$\frac{1}{2i} \lim_{c\to 0} \int_{W_o} \frac{uJ_{-u}(kr)J_u(ka)F(u)e^{cu^2}\,du}{Y_u(ka)\sin u\pi} = -\sum_{n=n_1}^{\infty} \frac{nJ_n(kr)J_n(ka)F(n)}{Y_n(ka)} . \tag{3.1}$$

In this equation, which will be established by closing the contour on the right hand side and taking the residues at the zeros of $\sin u\pi$, the path W_o, illustrated in Figure 1, consists of the parts of the rays $\arg u = \pm\frac{\pi}{4}$ lying to the right of the line $\mathrm{Re}(u) = n_1 - \frac{1}{2}$ together with the segment of this line cut off between the rays. The positive integer n_1 is chosen large enough to ensure that all of the zeros of $Y_u(ka)$ lie to the left of W_o. This choice is possible since the complex zeros u_n' are such that $\arg u_n' \to \pm\frac{\pi}{2}$ as $n \to \infty$.

The asymptotic form of $Y_u(x)$ as $u \to \infty$ in the half plane $\mathrm{Re}(u) > 0$ can be obtained from (2.12) and (2.14) which show that

$$Y_u(x) = -\frac{1}{\pi}(2/x)^u\Gamma(u)[1 + O(u^{-1})] \tag{3.2}$$

for large u and $|\arg u| \le \frac{\pi}{2} - \delta$. The derivative $Y_u'(x)$ can be obtained from (2.16) and (3.2) which yield the formula

$$xY_u'(x) = \frac{u}{\pi}(2/x)^u\Gamma(u)[1 + O(u^{-1})] . \tag{3.3}$$

On substituting (3.2), (3.3) into (2.11) it is found that

$$\pi R^2 \sin 2\theta \int_a^\infty r^{-1}|Y_u(kr)|^2 dr = \sinh(\pi R\sin\theta) + R\sin\theta|Y_u(ka)|^2[1 + O(R^{-1})] \tag{3.4}$$

whilst from (3.2), (2.14) it is seen that

$$|Y_u(ka)| = (2/\pi R)^{\frac{1}{2}} \exp[R\cos\theta\log(2R/kae) - R\theta\sin\theta][1 + O(R^{-1})] . \tag{3.5}$$

Equation (3.5) shows that the dominant term on the right hand side of (3.4) as $u \to \infty$ in the sector $|\theta| \le \frac{\pi}{2} - \delta$ is the second one so that, by (2.9),

$$\frac{F(u)}{Y_u(ka)} = O[|R\cos\theta|^{-\frac{1}{2}}] . \tag{3.6}$$

On combining this with (2.8), with (a,r) interchanged, it is seen that the integrand in (3.1) is

$$O\{|R\cos\theta|^{-\frac{1}{2}} \exp[-R\cos\theta\log(r/a) + cR^2\cos 2\theta]\}. \tag{3.7}$$

This applies in the sector $|\theta| \le \frac{\pi}{2} - \delta$ provided that u is large and bounded away from the positive integers.

It is seen from (3.7) with θ set equal to $\pm\frac{\pi}{4}$ therein that the integral in (3.1) is absolutely and uniformly convergent for all real c and the value of the limit may be deduced by setting $c = 0$ in the integrand. With the summability factor removed the bound (3.7) on the integrand reduces to $O\{R^{-\frac{1}{2}}\exp[-R\cos\theta\log(r/a)]\}$. The resulting integral can then be evaluated by closing the contour on the right hand side of W_o and taking the residues at the zeros of $\sin u\pi$ situated to the right of W_o. This procedure leads to the series on the right hand side of equation (3.1) which is therefore established.

To complete the transformation of I_2 it remains to deform the path W_o appearing in (3.1) onto the imaginary axis and take into account the residues at the poles of the integrand traversed in the process. These poles occur at the complex zeros u_n' as well as at those zeros of $\sin u\pi$ that are positioned between W_o and the imaginary axis. The latter zeros are at the points $u = 1, 2, \ldots, n_1 - 1$. This leads to the equation

$$\frac{1}{2i}\lim_{c\to 0}\int_L \frac{uJ_{-u}(kr)J_u(ka)F(u)e^{cu^2}\,du}{Y_u(ka)\sin u\pi} + \pi\lim_{c\to 0}\sum_{u_n'}\frac{uJ_{-u}(kr)J_u(ka)F(u)e^{cu^2}}{\sin u\pi\,(\partial/\partial u)Y_u(ka)}$$

$$+\lim_{c\to 0}\sum_{n=1}^{n_1-1}\frac{nJ_n(kr)J_n(ka)F(n)e^{cu^2}}{Y_n(ka)} = -\sum_{n=n_1}^{\infty}\frac{nJ_n(kr)J_n(ka)F(n)}{Y_n(ka)}$$

from which it follows

$$I_2 = -\pi\lim_{c\to 0}\sum_{u_n'}\frac{uJ_{-u}(kr)J_u(ka)F(u)e^{cu^2}}{\sin u\pi\,(\partial/\partial u)Y_u(ka)} - \sum_{n=1}^{\infty}\frac{nJ_n(kr)J_n(ka)F(n)}{Y_n(ka)}\,. \tag{3.8}$$

To establish the truth of the above equations the path W_o in (3.1) will be connected to the imaginary axis L by a sequence of paths C_n' which avoid the (complex) zeros of $Y_u(ka)$ and which recede to infinity as $n \to \infty$. The procedure to be followed here is similar to that described in the writer's earlier paper Naylor and Chang [1] where a slightly different integral was encountered. It was proved in Naylor and Chang [1], equation (39), or it can be deduced from (2.12) and (2.13), that

$$|Y_u(ka)| = -2(\pi u)^{-\frac{1}{2}}e^{-i\pi/4}\sinh(A+iB)[1+O(R^{-1})]$$

as $u\to\infty$ in $0<\delta\leq\theta\leq\frac{\pi}{2}$, where

$$A = R\cos\theta\log(2R/kae) - R\theta\sin\theta + \frac{1}{2}\log 2$$

$$B = R\sin\theta\log(2R/kae) + R\theta\cos\theta + \frac{\pi}{4}\,.$$

The large complex zeros $u_n' = Re^{i\theta}$ that lie in the first quadrant are given by the equations $A = 0$, $B = n\pi$ where n is a large positive integer. The zeros located in the fourth quadrant are situated at the conjugate points $Re^{-i\theta}$. The path C_n' may be taken as that whose polar equation in the sector $\frac{\pi}{2}-\delta\leq\theta\leq\frac{\pi}{2}$ is $B = (n+\frac{1}{2})\pi$ and which is continued beyond this sector by means of a circular arc to meet the path W_o. The part of C_n' located in the fourth quadrant is defined similarly. On the parts of C_n' lying inside the the stated sectors, $\sinh(A+iB)$ reduces to $\pm i\cosh A$ so that

$$|Y_u(ka)| = 2(\pi R)^{-\frac{1}{2}}\cosh A$$

$$= 2(\pi R)^{-\frac{1}{2}}\cosh[R\cos\theta\log(2R/kae) - R\theta\sin\theta + \frac{1}{2}\log 2] \tag{3.9}$$

as $R \to \infty$ in $\frac{\pi}{2} - \delta \le |\theta| \le \frac{\pi}{2}$. The asymptotic bound on $F(u)$ valid in these sectors is given by the formulas (42), (43) of Naylor and Chang [1]:

$$F(u) = O[|R^2 \sin 2\theta|^{-\frac{1}{2}} \exp|\tfrac{\pi}{2} R \sin\theta|]$$

$$+ O\{|R^2 \sin 2\theta|^{-\frac{1}{2}} \exp[R\cos\theta \log(2R/kae) - R\theta\sin\theta + \tfrac{1}{2}\log 2]\} \quad (3.10)$$

or

$$F(u) = O(\exp|\tfrac{\pi}{2} R \sin\theta|) \quad (3.11)$$

as $R \to \infty$ in the strip $|R\cos\theta| \le \frac{1}{3}$.

It follows on combining (3.9), (3.10) that

$$\frac{F(u)}{Y_u(ka)} = O[|R \sin 2\theta|^{-\frac{1}{2}} \exp|\tfrac{\pi}{2} R \sin\theta|]$$

for $\frac{\pi}{2} - \delta \le |\theta| < \frac{\pi}{2}$ and

$$\frac{F(u)}{Y_u(ka)} = O[R^{\frac{1}{2}} \exp|\tfrac{\pi}{2} R \sin\theta|]$$

in the strip $0 \le \mathrm{Re}(u) \le \frac{1}{3}$.

In the remaining sectors $\frac{\pi}{4} \le |\theta| \le \frac{\pi}{2} - \delta$ the formula (3.2) applies so that (3.6) is valid in this region also.

An inspection of the above bounds together with (2.8) reveals that the integrand appearing in (3.1) is, at most,

$$O\{R^{\frac{1}{2}}(r/a)^{-R\cos\theta} \exp[|\tfrac{\pi}{2} R\sin\theta| + cR^2\cos 2\theta]\} \quad (3.12)$$

as $R \to \infty$ in the sectors $\frac{\pi}{4} \le |\theta| \le \frac{\pi}{2}$. The expression (3.12) tends to zero as $R \to \infty$ sufficiently rapidly to permit the contour to be closed in the manner described. The dominant term in the exponential is the summability factor and $\cos 2\theta \le 0$ in the stated sectors.

This establishes the validity of (3.8) and on subtracting (2.25) and (3.8) we find that the second series in each formula cancels leaving the equation

$$I_1 - I_2 = \pi \sum_{u_n} \frac{uJ_{-u}(ka)J_u(kr)F(u)}{\sin u\pi\,(\partial/\partial u)Y_u(ka)}$$

$$+ \pi \lim_{c\to 0} \sum_{u_n'} \frac{uJ_{-u}(kr)J_u(ka)F(u)e^{cu^2}}{\sin u\pi\,(\partial/\partial u)Y_u(ka)}.$$

On substituting this expression for the integral present in (2.3) into the formula (2.1) and combining the two series involving the complex zeros u_n' with the aid of the identity (2.2), we obtain the formula

$$f(r) = \pi \sum_{u_n} \frac{uJ_{-u}(ka)J_u(kr)F(u)}{\sin u\pi\,(\partial/\partial u)Y_u(ka)}$$

$$+ \pi \lim_{c\to 0} \sum_{u_n'} \frac{uJ_u(kr)J_u(ka)F(u)\cot u\pi\, e^{cu^2}}{(\partial/\partial u)Y_u(ka)}. \quad (3.13)$$

Now it follows from (2.2), if u is a zero of $Y_u(ka)$, then $J_{-u}(ka) = J_u(ka)\cos u\pi$

and on making this change in the first of the series present in (3.13) we arrive at the expansion (1.6) quoted in the theorem.

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