Internat. J. Math. & Math. Sci.
Vol. 8 No. 4 (1985) 663-667

ON FIXED POINTS OF SET-VALUED DIRECTIONAL CONTRACTIONS

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(Received June 4, 1984 and in revised form April 25, 1985)

ABSTRACT. Using equivalent formulations of Ekeland's theorem, we improve fixed point theorems of Clarke, Sehgal, Sehgal-Smithson, and Kirk-Ray on directional contractions by giving geometric estimations of fixed points.

KEY WORDS AND PHRASES. l.s.c. function, (weak) directional contraction, fixed point, stationary point, Hausdorff psuedometric.
1980 AMS SUBJECT CLASSIFICATION CODES. 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

In [1], [2], Sehgal and Smithson proved fixed point theorems for set-valued weak directional contractions which extend earlier results of Clarke [3], Kirk and Ray [4], and Assad and Kirk [5]. In the present paper, results in [1], [2] are substantially strengthened by giving geometric estimations of locations of fixed points.

The following equivalent formulations [6] of the well-known central result of Ekeland [7], [8] on the variational principle for approximate solutions of minimization problems is used in the proofs of the main results.

THEOREM 1. Let (V, d) be a complete metric space, and $\phi : V \to R \cup \{+\infty\}$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Let $\varepsilon > 0$ and $\lambda > 0$ be given, and a point $u \in V$ such that

$$F(u) \le \inf_V F + \varepsilon.$$

Let $S(\lambda) = \{x \in V \mid F(x) \le F(u) - \varepsilon\lambda^{-1}d(u,x)\}$. Then the following equivalent conditions hold:

(i) There exists a point $v \in S(\lambda)$ satisfying

$$F(w) > F(v) - \varepsilon\lambda^{-1}d(v, w) \quad \text{for } \forall\ w \ne v.$$

(ii) If $T : S(\lambda) \to 2^V$ is a set-valued map satisfying the condition

$\forall\ x \in S(\lambda) \setminus T(x)\ \exists\ y \in V \setminus \{x\}$ such that

$$F(y) \le F(x) - \varepsilon\lambda^{-1}d(x, y),$$

then T has a fixed point $v \in S(\lambda)$.

(iii) If $f : S(\lambda) \to V$ satisfies

$$F(fx) \le F(x) - \varepsilon\lambda^{-1}d(x, fx)$$

for all $x \in S(\lambda)$, then f has a fixed point $v \in S(\lambda)$.

In Theorem 1, 2^V denotes the power set of V. Note that

$S(\lambda) \subset \{x \in V \mid F(x) \leq F(u),\ d(u, x) \leq \lambda\} \subset \bar{B}(u, \lambda)$

and $fS(\lambda) \subset S(\lambda)$, where $\bar{B}$ denotes the closed ball.

Throughout this paper, (V, d) denotes a metric space and $B(V)$ denotes the class of all nonempty bounded subsets of V with the Hausdorff pseudometric H defined by

$H(A, B) = \max\{\sup_{x \in A} d(x, B),\ \sup_{y \in B} d(y, A)\}$.

Also, $C(V)$ denotes the class of all nonempty compact subsets of V. For an $x \in V$ and $A \in C(V)$, we put

$$[x, A] = \{y \in A \mid d(x, y) = d(x, A)\},$$

which is nonempty. For x, $y \in V$, we denote

$$[x, y] = \{z \in V \mid d(x, z) + d(z, y) = d(x, y)\},$$

and

$$(x, y] = [x, y] \setminus \{x\},\ (x, y) = (x, y] \setminus \{y\}.$$

Let S be a nonempty subset of V and $T : S \to C(V)$ be a set-valued map. For $x \in S$ and $A \in C(V)$, the weak directional derivative $DT(x, y)$ of T at x in the direction of a $y \in [x, T(x)]$ is defined by

$$DT(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \inf\{\frac{H(Tx, Tz)}{d(x, z)} \mid z \in (x, y] \cap S\}, & \text{if } (x, y] \cap S \neq \phi, \\ \infty, & \text{if } (x, y] \cap S = \phi. \end{cases}$$

A map $T : S \to C(V)$ is called a weak directional contraction if there exists a $k \in [0, 1)$ such that for each $x \in S$, there exists a $y \in [x, T(x)]$ with $DT(x, y) < k$ [2].

A map $T : S \to B(V)$ is called a directional contraction if there exists a $k \in [0, 1)$ such that for each $x \in S$ and $y \in T(x)$,

$$H(T(z), T(x)) \leq kd(z, x)$$

for all $z \in [x, y] \cap S$ [7].

2. RESULTS.

THEOREM 2. Let S be a complete subset of V and $T : S \to C(V)$ a weak directional contraction for which the function $F(x) = d(x, T(x))$, $x \in S$, is l.s.c. Then for any $u \in S$ and $\varepsilon > 0$ satisfying $F(u) \leq (1 - k)\varepsilon$, T has a fixed point in $s(\varepsilon) \subset \bar{B}(u, \varepsilon) \cap S$.

PROOF. Choose a point $u \in S$ satisfying $F(u) \leq \inf_S F + (1 - k)\varepsilon$. Suppose $x \in S(\varepsilon) \setminus T(x)$. Since T is a weak directional contraction, there exists a $y \in [x, T(x)]$, $x \neq y$, with $DT(x, y) < k$. Hence, there exists a $z \in (x, y] \cap S$ such that

$$H(T(x), T(z)) < k\, d(x, z).$$

Since

$$d(x, z) + d(z, T(x)) \leq d(x, y) = d(x, T(x)),$$

we have

$$\begin{aligned} d(z, T(z)) &\leq d(z, T(x)) + H(T(x), T(z)) \\ &\leq d(x, T(x)) - d(x, z) + k\, d(x, z) \\ &\leq d(x, T(x)) - (1 - k)d(x, z). \end{aligned}$$

Hence, $F(z) \le F(x) - (1 - k)d(x, z)$. Therefore, by Therem 1(iii), T has a fixed point $v \in S(\varepsilon)$.

Theorem 2 is an improved version of Theorem (a) of [2] with much simpler proof. In fact, for suitable values of ε and k, the conclusion gives geometric estimations of locations of fixed points. However, for Theorem (b) of [2], such estimation seems to be hard to get.

Note also that for Theorem 1 of Clarke [3], we can apply our Theorem 2.

The following improves Corollary of [2] and a result of Kirk and Ray [4].

COROLLARY 1. Let S be a closed convex subset of a Banach space X and $T : S \to C(S)$ a map for which the function $F(x) = d(x, T(x))$, $x \in S$, is l.s.c. Suppose there exists a $k \in [0, 1)$ such that for each $x \in S$ there correspond a $y = y(x) \in [x, T(x)]$ and a $\delta \in (0, 1)$ satisfying

$$H(T(x), T(x + \delta(y - x)) \le k\delta||y - x||.$$

Then the conclusion of Theorem 1 follows.

PROOF. As in the proof in [2], T is a weak directional contraction with the conatant k.

THEOREM 3. Let S be a closed subset of a complete metric space V and $T : S \to B(V)$ a directional contraction with the constant α. If T satisfies

(a) for each $x \in S$, $y \in T(x) \setminus S$, there exists a $z \in (x, y) \setminus S$ with $T(z) \subset S$, and

(b) $g(x) = d(x, T(x))$ is l.s.c.,

then, for any $u \in S$, $\varepsilon > 0$ and β, $\alpha < \beta < 1$ satisfying $g(u) \le (1 - \beta)\varepsilon$, there exists a fixed point v of T in $S(\varepsilon) \cap S$.

LEMMA [4]. Under the hypothesis of Theorem 3, there exists a map $A : S \to B(X)$ with the following properties

i) for each $x \in S$, $A(x) \neq \phi$ and $A(x) \subset T(x)$

ii) if $y \in A(x)$, then $d(x, y) \le (1 - \beta + \alpha)^{-1}d(x, T(x))$,

iii) if $A(x) \cap S = \phi$ for some $x \in S$, then there exists a $y = y(x) \in A(x)$ and a $z = z(x, y) \in (x,y) \cap S$ such that

$$d(x, y) \le d(x, T(x)) + (\beta - \alpha)d(x, z). \tag{2.1}$$

PROOF OF THEOREM 3. Define a map $f : S \to S$ as follows: for $x \in S$ such that $A(x) \cap S \neq \phi$, let $f(x)$ be any element of $A(x) \cap S$; and for $x \in S$ such that $A(x) \cap S = \phi$, since there exist $y = y(x) \in A(x)$ and $z = z(x, y) \in (x, y)$ S satisfying (2.1) by Lemma, let $f(x) = z$. We claim that for any $x \in S$,

$$H(T(x), T(f(x))) \le \alpha\, d(x, f(x)). \tag{2.2}$$

This is clear if $A(x) \cap S = \phi$. If $A(x) \cap S \neq \phi$, since $f(x) \in T(x)$ and $f(x) \in [x, f(x)] \cap S$, the definition of T implies (2.2). Set $F(x) = (1 - \beta)^{-1}g(x)$. We know that for any $x \in S$ and $y = f(x)$,

$$F(y) \le F(x) - d(x, y)$$

holds as in the proof of [1, Theorem 1]. Therefore, by Theorem 1 (iii), for any $u \in S$ and $\varepsilon > 0$ satisfying $F(u) \le \inf_S F + \varepsilon$, there exists a fixed point v of f in $S(\varepsilon) \cap S$. This implies that $v \in T(v)$ for otherwise $f(v) \notin A(v) \cap S$ and hence by the definition of f, $A(v) \cap S = \phi$. Thus, $f(v) \in (v, y(v))$ for some $y(v) \in A(v)$. This contradicts $v \neq f(v)$. Consequently, $v \in T(v)$. Since $\inf_S F = 0$, u can be chosen so that $F(u) \leq \varepsilon$, that is, $d(u, T(u)) \leq (1 - \beta)\varepsilon$. This completes our proof.

Note that Theorem 3 is a strengthened form of [1, Theorem 1].

A metric space is said to be convex if for each $x, y \in X, x \neq y$, there exists a $z \in (x, y)$. It is known that if S is a closed subset of a complete convex metric space V and $x \in S$ and $y \notin S$, then there exists a $z \in [x, y) \cap \delta S$ where δ is the boundary.

Now, we obtain the following improved version of [1, Corollary 1] as an immediate consequence of Theorem 3.

COROLLARY 2. Let S be a closed subset of a complete convex metric space V. Let $T : S \to B(V)$ be a directional contraction which the constant α such that $T(\delta S) \subset S$. If $g(x) = d(x, T(x))$ is l.s.c. on S, then for any $u \in S$, $\varepsilon > 0$, and β, $\alpha < \beta < 1$, satisfying $g(u) \leq (1 - \beta)\varepsilon$, there exists a fixed point v of T in $S(\varepsilon) \cap S$.

Also, the following improves [1, Corollary 2] and an earlier result of Assad-Kirk [5].

COROLLARY 3. Let S be a closed subset of a complete convex metric space V. Suppose $T : S \to B(X)$ is a contraction, that is, there exists an $\alpha \in [0, 1)$ such that for all $x,y \in S$,

$$H(T(x), T(y)) \leq \alpha\, d(x, y).$$

If $T(\delta S) \subset S$, then for any $u \in S$, $\varepsilon > 0$, and β, $\alpha < \beta < 1$, satisfying $d(u, T(u)) \leq (1 - \beta)\varepsilon$, either u is a fixed point of T or there exists a fixed point v of T in

$$S(\varepsilon) \cap S \setminus B(u, s)$$

where $s = d(u, T(u))\,(1 + \alpha)^{-1}$.

PROOF. Since a contraction is a directional contraction and $g(x) = d(x, T(x))$ is continuous, by Corollary 2, T has a fixed point $v \in S(\varepsilon) \cap S$. Suppose u is not fixed under T. Then for any $y \in B(u, s) \cap S$ we have

$$d(u, T(u)) \leq d(u, y) + d(y, T(u))$$
$$< s + d(y, T(u)),$$

that is,

$$\alpha(1 + \alpha)^{-1} d(u, T(u)) < d(y, T(u)).$$

Hence,

$$d(y, T(u)) > \alpha s > \alpha d(y, u).$$

Suppose $y \in T(y)$. Then we have

$$H(T(y), T(u)) > \alpha d(y, u),$$

a contradiction. This completes our proof.

ACKNOWLEDGEMENT: The author expresses his gratitude to the referee for his kind comments.

This work was partially supported by a grant from the Korea Science and Engineering Foundation, 1983-84.

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