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ON A FUNCTION RELATED TO RAMANUJAN'S TAU FUNCTION

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ABSTRACT. For the function $\psi = \psi_{12}$, defined by $\sum_1^\infty \psi(n)x^n = x\prod_1^\infty (1-x^{2n})^{12}$ $(|x|<1)$, the author derives two simple formulas. The simpler of these two formulas is expressed solely in terms of the well-known sum-of-divisors function.

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1. INTRODUCTION.

Following Ramanujan [4,p. 155] we define for each positive divisor α of 24 an arithmetical function ψ_α as follows:

$$\sum_1^\infty \psi_\alpha(n)x^n = x\prod_1^\infty (1-x^{24n/\alpha})^\alpha, \tag{1.1}$$

an identity which is valid for each complex number x such that $|x| < 1$. Of course, $\psi_{24} = \tau$, the celebrated Ramanujan tau fuction. In this paper we are specifically concerned with ψ_{12}($=\psi$ for simplicity). As a matter of fact, we derive two explicit formulas for ψ. Since these formulas involve the sum-of-divisors function and the counting function for sums of eight squares, we need the following definition.

Definition. (i) For each positive integer n, $\sigma(n)$ denotes the sum of all positive divisors of n. (ii) for each nonnegative integer n, $r_k(n)$ denotes the cardinality of the set

$$\{(x_1,x_2,\ldots,x_k) \in \mathbb{Z}^k \mid n = x_1^2 + x_2^2 + \ldots + x_k^2\},$$

k an arbitrary positive integer.

We can now state our main result.

Theorem 1. For each nonnegative integer m,

$$\psi(2m+1) = \sum_{i-0}^{m} (-1)^i r_8(i)\sigma(2m-2i+1), \tag{1.2}$$

$$\psi(2m+2) = 0. \tag{1.3}$$

In section 2 we prove theorem 1, and thereafter prove a corollary which gives a formula expressing ψ solely in terms of σ.

2. PROOF OF THEOREM 1.

Our proof requires the following three identities, each of which is valid for each complex number x such that $|x| < 1$.

$$\prod_1^\infty (1+x^n)(1-x^{2n-1}) = 1 \tag{2.1}$$

$$\prod_{1}^{\infty} (1-x^n)(1-x^{2n-1}) = \sum_{-\infty}^{\infty} (-x)^{n^2} \tag{2.2}$$

$$\prod_{1}^{\infty} (1-x^{2n})(1+x^n) = \sum_{0}^{\infty} x^{n(n+1)/2} \tag{2.3}$$

Identity (2.1) is due to Euler, while (2.2) and (2.3) are due to Gauss. For proofs see [3, pp. 277-284]. We also need a fourth identity which the author has not been able to locate in the literature. This we here record in the following lemma.

LEMMA. For each complex number x such that $|x| < 1$,

$$\{\sum_{0}^{\infty} x^{m(m+1)/2}\}^4 = \sum_{0}^{\infty}\sigma(2m+1)x^m \tag{2.4}$$

Proof: Here we need the following two identities, stated and proved in [1, p. 313].

$$\prod_{1}^{\infty}(1-x^{2n})^2(1+x^{2n-1})^4 = \{\sum_{-\infty}^{\infty} x^{2m^2}\}^2 + x\{\sum_{-\infty}^{\infty} x^{2m(m+1)}\}^2$$

$$\prod_{1}^{\infty}(1-x^{2n})^2(1-x^{2n-1})^4 = \{\sum_{-\infty}^{\infty} x^{2m^2}\}^2 - x\{\sum_{-\infty}^{\infty} x^{2m(m+1)}\}^2$$

We square these identities, add the resulting identities, and utilize the fact that the fourth power of the right side of (2.2) generates $(-1)^n r_4(n)$, to write:

$$2\sum_{0}^{\infty} r_4(2n)x^{2n} = \sum_{0}^{\infty} r_4(n)x^n + \sum_{0}^{\infty} (-1)^n r_4(n)x^n$$

$$= 2\sum_{0}^{\infty} r_4(n)x^{2n} + 2x^2\{\sum_{-\infty}^{\infty} x^{2m(m+1)}\}^4,$$

whence

$$x^2\{\sum_{-\infty}^{\infty} x^{2m(m+1)}\}^4 = \sum_{0}^{\infty}[r_4(2n)-r_4(n)]x^{2n}$$

$$= \sum_{0}^{\infty}[r_4(4m)-r_4(2m)]x^{4m}$$

$$+ \sum_{0}^{\infty}[r_4(4m+2) - r_4(2m+1)]x^{4m+2}$$

$$= \sum_{0}^{\infty}[24\sigma(2m+1) - 8\sigma(2m+1)]x^{4m+2}$$

$$= 2^4 \sum_{0}^{\infty} \sigma(2m+1)x^{4m+2}$$

Here, we've made use of Jacobi's formula for $r_4(n)$. Now, cancelling 2^4x^2 and subsequently letting $x \to x^{1/4}$, we obtain (2.4).

Continuing with the proof of theorem 1, we use (2.1) to rewrite (2.3) as

$$\prod_{1}^{\infty}(1-x^n)(1-x^{2n-1})^{-2} = \sum_{0}^{\infty} x^{n(n+1)/2}$$

We then raise the identity to the fourth power, and multiply the resulting identity by the eighth power of identity (2.2) to get

$$\prod_{1}^{\infty}(1-x^n)^{12} = \{\sum_{-\infty}^{\infty} (-x)^{n^2}\}^8 \{\sum_{0}^{\infty} x^{n(n+1)/2}\}^4$$

$$= \sum_{i=0}^{\infty} (-1)^i r_8(i)x^i \cdot \sum_{j=0}^{\infty} \sigma(2j+1)x^j$$

$$= \sum_{n=0}^{\infty} x^n \sum_{i=0}^{n} (-1)^i r_8(i)\sigma(2n-2i+1).$$

In the foregoing we then let $x \to x^2$, and multiply the resulting identity by x to get

$$\sum_1^\infty \psi(n)x^n = x.\prod_1^\infty (1-x^{2n})^{12}$$

$$= \sum_0^\infty x^{2m+1} \sum_0^m (-1)^i r_8(i)\sigma(2m-2i+1)$$

Comparing coefficients of x^n we thus prove our theorem.

By appeal to the well-known formula for r_8, viz.,

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3, \quad n \in \mathbb{Z}^+$$

(e.g., see [3, p. 314]), we eliminate r_8 from (1.2) as follows:

$$\psi(2m+1) = \sigma(2m+1) + 16 \sum_{i=1}^m \sigma(2m-2i+1) \sum_{d|i} (-1)^d d^3$$

In order to extend the inner sum over all d in the range $1,2,\dots,i$ we define $\varepsilon(i,d)$ to be 1, if d divides i, to be 0, otherwise. Hence,

$$\psi(2m+1) = \sigma(2m+1) + 16 \sum_{i=1}^m \sum_{d=1}^i (-1)^d \sigma(2m-2i+1)\varepsilon(i,d)d^3$$

$$= \sigma(2m+1) + 16 \sum_{d=1}^m (-1)^d d^3 \sum_{i=d}^m \varepsilon(i,d)\sigma(2m-2i+1)$$

$$= \sigma(2m+1) + 16 \sum_{d=1}^m (-1)^d d^3 \sum_{k=1} \sigma(2m-2kd+1)$$

The upper limit of summation of the sum indexed by k is naturally $[m/d]$, the integral part of m/d. Thus, we have proved the following

COROLLARY. For each nonnegative integer m,

$$\psi(2m+1) = \sigma(2m+1) + 16 \sum_{d=1}^m (-1)^d d^3 \sum_{k=1}^{m/d} \sigma(2m-2kd+1).$$

CONCLUDING REMARKS. According to Hardy, Ramanujan conjectured that each of the ψ_α (for α dividing 24) is multiplicative; e.g., see [2, p. 184]. These conjectures were later confirmed by L. J. Mordell. Owing to classical identities of Euler and Jacobi, ψ_1 and ψ_3 are trivially defined. Ramanujan himself deduced formulas for ψ_2, ψ_4, ψ_6 and ψ_8.

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