

RELATIVE INTEGRAL BASIS FOR ALGEBRAIC NUMBER FIELDS

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ABSTRACT. At first conditions are given for existence of a relative integral basis for $O_K \cong O_k^{n-1} \oplus I$ with $[K;k] = n$. Then the construction of the ideal I in $O_K \cong O_k^{n-1} \oplus I$ is given for proof of existence of a relative integral basis for $O_{K_4}(\sqrt{m_1}, \sqrt{m_2})/O_k(\sqrt{m_3})$. Finally existence and construction of the relative integral basis for $O_{K_6}(\sqrt[3]{n}, \sqrt{-3})/O_{K_3}(\sqrt[3]{n}), O_{K_6}(\sqrt[3]{n}, \sqrt{-3})/O_{K_2}(\sqrt{-3})$ for some values of n are given.

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1. INTRODUCTION

Throughout this article the following notation will be used:

- Q: field of rational numbers
- Z: rational integers
- k, K ($Q \subseteq k \subseteq K$): algebraic number fields
- disc(x): discriminant of element x
- $D_{K/k}$: relative different of extension K/k
- $O_{k_i} = O_i$: ring of integers of k_i
- h_k : class number for k
- P.I.: principal ideal
- $N_{K/k}a$: relative norm of an ideal a in K for extension K/k .

2. FINITELY GENERATED MODULES

In [1, p. 24] it was shown that if M is a finitely generated module over a Dedekind ring R then

$$M \cong R^m \oplus A \oplus I, \tag{2.1}$$

where I is an ideal in R , A is a torsion-submodule and m is a positive integer.

Now for extension K/k with $[K;k] = n$, by (2.1) we have

$$O_K \cong O_k^{n-1} \oplus I \tag{2.2}$$

so by this we have:

THEOREM 2.3. In the extension K/k for $[K;k] = n$, O_K has relative integral basis over $O_k \iff I = \text{P.I.}$

ILLUSTRATION 2.4. Let $k_1 = \mathbb{Q}(\sqrt{2})$, $k_2 = \mathbb{Q}(\sqrt{-7})$. Does a relative integral basis of $\mathbb{O}(\sqrt{2}, \sqrt{-7})/\mathbb{O}_3 = \mathbb{O}(\sqrt{-14})$ exist? see also [2].

SOLUTION. By (2.2), a "relative integral basis" exists $\Leftrightarrow I = \text{P.I.}$, otherwise not.

We will construct an ideal I in \mathbb{O}_{k_3} where $\mathbb{O}_K \cong \mathbb{O}_{k_3}^{2-1} \oplus I$. Since $(d_{K_1}, d_{K_2}) = (2 \cdot 4, -7) = 1$, then using a theorem given in [3, p. 218],

$$\begin{aligned} \mathbb{O}_K &= \left[1, \sqrt{2} \right] \times \left[1, \frac{1+\sqrt{-7}}{2} \right] \cdot z = \left[1, \sqrt{2}, \frac{1+\sqrt{-7}}{2}, \frac{\sqrt{2}+\sqrt{-14}}{2} \right] \cdot z, \\ \mathbb{O}_K &= \left[1, \sqrt{-14}, \frac{1+\sqrt{-7}}{2}, \frac{\sqrt{2}+\sqrt{-14}}{2} \right] \cdot z = \left[1, \sqrt{-14} \right] \oplus R, \end{aligned}$$

where R is an \mathbb{O}_K -module,

$$R = \left[\frac{1+\sqrt{-7}}{2} + s + t\sqrt{-14}, \frac{\sqrt{2}+\sqrt{-14}}{2} + u + v\sqrt{-14} \right], \quad (2.5)$$

$\sqrt{-14} R \subseteq R$.

$$R \cong \sqrt{-14} R = \left[\frac{\sqrt{-14} + -7\sqrt{2}}{2} + s\sqrt{-14} + -14t, -7 + \sqrt{-7} + u\sqrt{-14} + -14v \right] \quad (2.6)$$

We take (2.5) and (2.6) proportional; then

$$\begin{cases} 1 + \sqrt{-7} + 2s + 2t\sqrt{-14} = -7 + \sqrt{-7} + u\sqrt{-14} + -14 \cdot v \\ 2s + 2t\sqrt{-14} = -8 + u\sqrt{-14} + -14v \\ u = 2t, \quad s = -8 + -14v, \quad \text{for } u = v = t = 0, \quad s = -4, \end{cases}$$

$$\begin{cases} \sqrt{-14} + -7\sqrt{2} + 2s\sqrt{-14} - 28t = (\sqrt{2} + \sqrt{-7} + 2u + \sqrt{-14}) \cdot -7 \\ -28t + \sqrt{-14} (1+2s) = -14u + -7\sqrt{-14} (1 + 2v) \\ u = 2t, \quad 1+2s = -7(1+2v), \quad \text{for } u = t = v = 0, \quad s = -4. \end{cases}$$

Then,

$$\begin{aligned} R &= \left[\frac{1+\sqrt{-7}}{2} - 4 + 0, \frac{\sqrt{2}+\sqrt{-14}}{2} \right] = \left[\frac{-7 + \sqrt{-7}}{2}, \frac{\sqrt{2} + \sqrt{-14}}{2} \right] \\ \sqrt{-14} R &= \left[\frac{-7\sqrt{2} - 7\sqrt{-14}}{2}, \frac{2\sqrt{-7} - 14}{2} \right] = \left[\frac{17(\sqrt{2} + \sqrt{-14})}{2}, -7 + \sqrt{-7} \right], \\ R &= \frac{\sqrt{-14} (1+\sqrt{-7})}{2\sqrt{2}} \cdot [2, \sqrt{-14}], \quad R \cong I = [2, \sqrt{-14}] \text{ is an ideal in } \mathbb{O}_K = [1, \sqrt{-14}] \cdot z. \end{aligned}$$

Since $I = [2, \sqrt{-14}]$ is not P.I. in \mathbb{O}_3 , then \mathbb{O}_K does not have a relative integral over \mathbb{O}_3 . The ideal $I = [2, \sqrt{-14}]$ is unique (up to equivalence of ideals).

The method of the previous theorem is only good for the case $n = 2$ since for $n \geq 3$, computation of an ideal in $\mathbb{O}_k \cong \mathbb{O}_k^{n-1} \oplus I$ is too difficult. Thus we need a relation such as the following between I and one of the invariants in the extension K/k .

THEOREM 2.7. If C is the class of ideals in k containing $d_{K/k}$ and $C_{K/k}$ is a class containing I , then

$$C = C_{K/k}^2.$$

Now we will give the "criterion for existence of a relative integral basis", for the extension K/k . See Norkiewicz et al. [1,4,5,6].

THEOREM 2.8. Let $[K:k] = n$, $h_k = \text{odd}$, then \mathbb{O}_K has a "relative integral basis" over $\mathbb{O}_k \Leftrightarrow d_{K/k}$ (relative discriminant) is a principal ideal. For more details see [1, p. 359].

PROOF. \Rightarrow : If O_K/O_k has a relative integral basis, $I = P.I.$ Therefore by Theorem 2.7 $d_{K/k}$ is P.I.

\Leftarrow : $d_{K/k} = P.I.$, so every ideal in the class of $C_{K/k}^2$ is P.I. Therefore $I^2 = P.I.$, since $(2, h_k) = 1$. Then $I = P.I.$ and by (2.2), O_K has a relative integral basis over O_k .

COROLLARY 2.9. If $O_k = P.I.D.$, then $h_k = 1$ is odd and $d_{K/k} = P.I.$ Thus by Theorem 2.8 for every finite extension of k where the ring of integers is its P.I.D., a relative integral basis exists.

ILLUSTRATION 2.10. Let $k_3 = Q(\sqrt[3]{213})$ and $k = Q(\sqrt{-3})$, $K_6 = Q(\sqrt{-3}, \sqrt[3]{213})$, $h_3 = 21$. Does a relative basis of O_{K_6}/k_3 exist or not?

We know that for $n=ab^2$, $(a,b) = 1$, $ab \neq 1$ in $k_3 = Q(\sqrt[3]{n})$,

$$O_3 = \left[1, \sqrt[3]{ab^2}, \sqrt[3]{a^2b} \right] \cdot z \text{ for } a \not\equiv \pm b \pmod{9}, \text{ and}$$

$$O_3 = \left[1, \frac{\sqrt[3]{ab^2} + \sqrt[3]{a^2b}}{3}, \sqrt[3]{ab^2}, \sqrt[3]{a^2b} \right] \cdot z \text{ for } a \equiv b \pmod{9}.$$

We call these two cases respectively Type I and Type II. $3 = (\sqrt{-3})^2$ in k_2 .

In Type I, $3 = 3_{11}^6$, $3_{11}^2 = 3_1$, $3_{11}^3 = \sqrt{-3}$, so $3 = 3_1^3$ and we define $f_0 = 3ab$.

In Type II, $3 = 3_{11}^2 3_{12}^2 3_{13}^2$, $3_{11}^2 = 3_2$, $3_{12} \cdot 3_{13} = 3_1$, $3_{11} \cdot 3_{12} \cdot 3_{13} = \sqrt{-3}$, so

$3 = 3_1^2 \cdot 3_2$ and we define $f_0 = ab$. $d_{6/3} = 3_{11}$, $d_{6/3} = 3_{11}^2 = 3_1$, $d_{6/3} = 3_1 =$

$(-3, \sqrt[3]{213})$. See [5, p. 221]. By Theorem 2.5, since h_3 is odd and

$d_{6/3} = (-3, \sqrt[3]{213} - 6) = (-343, \sqrt[3]{213} - 6) = (\sqrt[3]{213} - 6)$ is a P.I., so a relative integral basis exists.

Incidentally in (3.1) we will prove that if $3 \nmid h_3$ then O_6 has a relative integral basis over O_3 , but here $h_3 = 21$ so $3|h_3$ and also a relative integral basis exists.

3. EXISTENCE OF A RELATIVE INTEGRAL BASIS:

BY SOME CONDITIONS ON n FOR $O_6(\sqrt[3]{n}, \sqrt{-3}) / O_3(\sqrt[3]{n})$.

Now here we will show that for some $n \in \mathbb{Z}$ this extension has relative integral basis.

THEOREM 3.1. If $3 \nmid h_3$, then O_{K_6} has relative integral basis over O_3 for Type 1.

PROOF. By Theorem in [7, p. 222], O_6 has a relative integral basis over $O_3 \Leftrightarrow d_{6/3} / \sqrt{-3} = 3_{11} / \sqrt{-3} = 1/3_1$ is a P.I. in O_6 generated by an element of k_3 . But $3_1 = (-3, \sqrt[3]{n} + 1)$ when $3 \nmid ab$ and $3_1 = (-3, \sqrt[3]{n})$ when $3|ab$ in Type I and $(3) = (3_1 \cdot 3_2^2)$ for Type 2.

Now, $3 \nmid h_3$ so $(3_1)^3 = (-3, \sqrt[3]{n} \pm 1)^3 = (3)$ or $(3_1)^3 = (-3, \sqrt[3]{n})^3 = (3)$ for P.I., so: $3_1 = (-3, \sqrt[3]{n} \pm 1)$ or $(-3, \sqrt[3]{n}) = 3_1$ is P.I. Then $1/3_1$ also generates a P.I. in O_6 . In Type II, $(3) = (3_1 \cdot 3_2^2)$, it is dependent on ideals 3_1 and 3_2 ; therefore in this case, a relative basis may exist or may not exist.

But it may be that $3|h_3$ and again O_6 has relative basis over O_3 . For example in $k_3 = Q(\sqrt[3]{213})$, $h_3 = 21$ and $3_1 = (\sqrt[3]{213} - 6)$, so $3|21$ and a relative integral basis exists. Therefore the inverse of Theorem 3.1 is not true in general.

Next we show in [8] for $k_3 = Q(\sqrt[3]{n})$:

THEOREM 3.2. One of the following statements holds:

- $$3 \nmid h_3 \Leftrightarrow \begin{cases} 1) & n = 3 \\ 2) & n = p, \quad p = \text{prime}, \quad p \equiv -1 \pmod{3} \\ 3) & n = 3p \text{ or } 9p, \quad p = \text{prime} \equiv 2 \text{ or } 5 \pmod{9} \\ 4) & n = p \cdot q \text{ (} p, q \text{ are primes), } \quad p \equiv 2 \text{ and } q \equiv 5 \pmod{9} \\ 5) & n = p^2 \cdot q \text{ (} p, q \text{ are distinct primes), } \quad p \equiv q \equiv 2 \text{ or } 5 \pmod{9}. \end{cases}$$

DEFINITION 3.3. A number n is called a Honda number if it is a number in the table for Theorem 3.2.

By Theorems 3.1 and 3.2 we have:

THEOREM 3.4. If n is a Honda number in type I, $O_6(\sqrt[3]{n}, \sqrt{-3})$ necessarily has a relative integral basis over $O_3(\sqrt[3]{n})$.

4. RELATIVE INTEGRAL BASIS OF $O_{K_6}(\sqrt[3]{n}, \sqrt{-3}) / O_{K_3}(\sqrt[3]{n})$.

We proved in Theorem 3.1 that if $3 \nmid h_3$, then 3_1 is P.I. only for Type I. Therefore a relative integral basis for O_6/O_3 exists, since by the theorem in

[3, p. 201], $\text{disc} \left(1, \frac{3+\sqrt{-3}}{2 \cdot 3_1} \right) = d_{K_6/K_3} = 3_1$. Therefore $\left[1, \frac{3+\sqrt{-3}}{2 \cdot 3_1} \right]$ is a relative integral basis for O_6 over O_3 , so:

$$O_{K_6} = \left[1, \frac{3+\sqrt{-3}}{2 \cdot 3_1} \right] \cdot O_{K_3}.$$

5. CONSTRUCTION OF RELATIVE INTEGRAL BASIS FOR $O_6(\sqrt[3]{n}, \sqrt{-3})/O_2(\sqrt{-3})$.

Since $O_2(\sqrt{-3})$ is P.I.D., then by (2.3) $O_6(\sqrt[3]{n}, \sqrt{-3})/O_2(\sqrt{-3})$ has a relative integral basis.

THEOREM 5.1. Let $\lambda = \frac{3_1}{2(3+\sqrt{-3})}$. For Type I and 3_1 is a P.I., if $3|a$ then

$$O_6 = \left[1, \frac{\sqrt[3]{ab^2}}{\lambda}, \frac{\sqrt[3]{a^2b}}{\lambda^2} \right] \cdot O_2,$$

and if $3|b$ then

$$O_6 = \left[1, \frac{\sqrt[3]{ab^2}}{\lambda^2}, \frac{\sqrt[3]{a^2b}}{\lambda} \right] \cdot O_2.$$

PROOF. Since $N_{6/2}(\sqrt[3]{ab^2/\lambda})$ and $N_{6/2}(\sqrt[3]{a^2b/\lambda^2})$ are in O_2 , then are integers.

If $d_{6/2} = \text{disc}(1, \sqrt[3]{ab^2/\lambda}, \sqrt[3]{a^2b/\lambda^2}) \cdot O_2$, then by the theorem in [3, p. 201] $x = [1, \sqrt[3]{ab^2/\lambda}, \sqrt[3]{a^2b/\lambda^2}]$ is a relative integral basis of O_6/O_2 , so we are going to compute $\text{disc}(x)$.

$$\begin{aligned} \text{disc } x &= \begin{vmatrix} 1 & \sqrt[3]{ab^2/\lambda} & \sqrt[3]{a^2b/\lambda^2} \\ 1 & \rho \sqrt[3]{ab^2/\lambda} & \rho^2 \sqrt[3]{a^2b/\lambda^2} \\ 1 & \rho^2 \sqrt[3]{ab^2/\lambda} & \rho \sqrt[3]{a^2b/\lambda^2} \end{vmatrix}^2 \\ &= (ab)^2 \cdot \frac{(3+\sqrt{-3})^6}{3^4 \cdot 2^6} \begin{vmatrix} 3_1^2 & 3_1 & 1 \\ 3_1^2 & 3_1 \rho & \rho^2 \\ 3_1^2 & 3_1 \rho^2 & \rho \end{vmatrix}^2 \end{aligned}$$

$$= \frac{(ab)^2 \cdot (3+\sqrt{-3})^6}{3^4 \cdot 2^6} \left\{ 3_1^2 (3_1^{\rho^2} - 3_1^{\rho}) + 3_1^{\rho^2} (3_1^{\rho^2} - 3_1^{\rho}) + 3_1^{\rho^2} (3_1^{\rho^2} - 3_1^{\rho}) \right\}^2$$

$$= 3^2 \cdot a^2 b^2 .$$

For "Type I" we have $d_{6/3} = 1_0^2 = \text{disc } x$, so $x = [1, \sqrt[3]{ab^2}/\lambda, \sqrt[3]{a^2b}/\lambda^2]$ is a relative integral basis of O_6/O_3 . See Cohn et al. [7,9,10].

ILLUSTRATION 5.2. For $K_3 = Q(\sqrt[3]{213})$, the ideal $3_1 = (\sqrt[3]{213} - 6)$ is P.I. and $3|ab^2$ (Type I, $3|a$), so

$$O_{K_6} = \left[1, \frac{\sqrt[3]{213}}{\lambda}, \frac{\sqrt[3]{213}^2}{\lambda^2} \right] \cdot O_2, \text{ where } \lambda = \frac{\sqrt[3]{213} - 6}{\frac{1}{2}(3+\sqrt{-3})} .$$

We have to mention that if 3_1 is not a P.I., this is still an open question.

THEOREM 5.3. Assume $k_3 = Q(\sqrt[3]{ab^2})$, $(3) = 3_1^3 = (\sqrt[3]{ab^2} \pm n)^3$, for $3 \nmid ab$ and "Type I", then:

$$O_{K_6} = \left[1, \frac{\sqrt[3]{ab^2} + n - \sqrt{-3}}{3_1}, \frac{\sqrt[3]{ab^2} + \sqrt[3]{a^2b} + t_1}{3_1} \right] O_{k_2}, \tag{5.4}$$

where $t_1 = 0$ for $a = 3k+1$ and $b = 3k+2$ or conversely and $t_1 = 1$ for $a = b = 3k+1$ and $t_1 = -1$ for $a = b = 3k+2$.

PROOF. Now

$$\frac{\sqrt[3]{ab^2} + n - \sqrt{-3}}{3_1} = \alpha_1, \quad \frac{\sqrt[3]{ab^2} + \sqrt[3]{a^2b} + t_1}{3_1} = \alpha_2$$

are integrals because $N_{K_6/K_2}(\alpha_1)$ and $N_{K_6/K_2}(\alpha_2)$ are integers. We take $x = [1, \alpha_1, \alpha_2]$ and $t_2 = +n - \sqrt{-3}$:

$$\text{disc}(x) = \begin{vmatrix} 1 & \frac{\sqrt[3]{ab^2} + t_2}{3_1} & \frac{\sqrt[3]{ab^2} + \sqrt[3]{a^2b} + t_1}{3_1} \\ 1 & \rho \frac{\sqrt[3]{ab^2} + t_2}{3_1^{\rho}} & \rho \frac{\sqrt[3]{ab^2} + \rho^2 \sqrt[3]{a^2b} + t_1}{3_1^{\rho}} \\ 1 & \rho^2 \frac{\sqrt[3]{ab^2} + t_2}{3_1^{\rho^2}} & \rho^2 \frac{\sqrt[3]{ab^2} + \rho \sqrt[3]{a^2b} + t_1}{3_1^{\rho^2}} \end{vmatrix}^2$$

$$= \frac{1}{3^2} \begin{vmatrix} 3_1 & \sqrt[3]{ab^2} + t_2 & \sqrt[3]{a^2b} + t_3 \\ 3_1^{\rho} & \rho \sqrt[3]{ab^2} + t_2 & \rho^2 \sqrt[3]{a^2b} + t_3 \\ 3_1^{\rho^2} & \rho^2 \sqrt[3]{ab^2} + t_2 & \rho \sqrt[3]{a^2b} + t_3 \end{vmatrix}^2$$

for $t_3 = t_1 - t_2$.

$$\text{disc}(x) = \left[3_1 \cdot (\rho^2 ab + \rho t_3 \sqrt[3]{ab^2} + t_2 \rho \sqrt[3]{a^2b} + t_2 \cdot t_3 - \rho ab - \rho^2 t_2 \cdot \sqrt[3]{a^2b} - t_3 \rho^2 \sqrt[3]{ab^2} - t_2 \cdot t_3) \right. \\ \left. + 3_1^{\rho} \cdot (\rho^2 ab + t_2 \sqrt[3]{a^2b} + t_3 \rho^2 \sqrt[3]{ab^2} + t_2 \cdot t_3 - \rho ab - t_3 \sqrt[3]{ab^2} - t_2 \cdot \sqrt[3]{a^2b} - t_2 \cdot t_3) \right. \\ \left. + 3_1^{\rho^2} \cdot (\rho^2 ab + t_3 \sqrt[3]{ab^2} + t_2 \rho^2 \sqrt[3]{a^2b} + t_2 \cdot t_3 - \rho ab - t_2 \sqrt[3]{a^2b} - \rho t_3 \sqrt[3]{ab^2} + t_2 \cdot t_3) \right] \cdot \frac{1}{3^2}$$

$$\text{disc}(x) = \left[+ 3nab\rho^2 + 3nab\rho + 3ab\rho t_2 - 3ab\rho^2 t_2 \right] \cdot \frac{1}{3^2}$$

For $t_2 = +n - \sqrt{-3}$ we have

$$\text{disc}(x) = \left[3nab(\rho^2 - \rho) - 3abt_2(\rho^2 - \rho) = 3ab(\rho^2 - \rho)(n - t_2) = 3ab(\rho^2 - \rho)(n - n + \sqrt{-3}) \right] \cdot \frac{1}{3^2}$$

$$\text{disc}(x) = 1/3^2 \cdot 3^2 \cdot a^2 b^2 \cdot 3^2$$

$$\text{disc}(x) = 3^2 \cdot a^2 b^2, \text{ since } d_{K_6/K_2} = f_0^2 = (3ab)^2 = \text{disc}(x).$$

Thus (5.4) is a relative integral basis for O_{K_6} over O_{K_2} .

Also we have the same result for the case $t_2 = -n - \sqrt{-3}$.

ILLUSTRATION 5.5. (1) If $k_3 = Q(\sqrt[3]{2})$, $3_1 = (\sqrt[3]{2} + 1)$ then $n = 1$ and $a = 3k+2$, $b = 3k+1$, so $t_1 = 0$. Therefore:

$$(1) \quad O_{K_6} = \left[1, \frac{\sqrt[3]{2} + \frac{1}{3} - \sqrt{-3}}{3_1}, \frac{\sqrt[3]{2} + \frac{3}{4} + 0}{3_1} \right] \cdot O_{K_2}.$$

(2) For $k_3 = Q(\sqrt[3]{5})$, $3_1 = (\sqrt[3]{5} - 2)$, $n = -2$ and $t_1 = 0$. We have

$$O_{K_6} = \left[1, \frac{\sqrt[3]{3} - 2 - \sqrt{-3}}{3_1}, \frac{\sqrt[3]{5} + \frac{3}{\sqrt[3]{25}} + 0}{3_1} \right] \cdot O_{K_2}.$$

For all Honda numbers 3_1 is necessarily P.I. so for such n we can construct a relative integral basis as in (5.4) for O_{K_6}/K_2 .

THEOREM 5.6. The relative integral basis in "Type II" of O_{K_6} over O_{K_2} is:

$$O_{K_6} = \left[1, \frac{\sqrt[3]{ab^2 - 1}}{\sqrt{-3}}, \frac{1 + \sqrt[3]{ab^2} + \sqrt[3]{a^2 b}}{3} \right] \cdot O_{K_2}.$$

PROOF. For "Type II" (i.e. $a \equiv \pm b \pmod{9}$), $\theta_0 = \frac{1 + \sqrt[3]{ab^2} + \sqrt[3]{a^2 b}}{3}$ satisfies in equation $\theta_0^3 - \theta_0^2 + \theta_0 \cdot \frac{1-ab}{9} - \frac{1-a+a^2b+ab^2}{27} = 0$, so then it is an integral and also $(\sqrt[3]{ab^2 - 1})/\sqrt{-3}$ is an integral, because: From $(\sqrt[3]{ab^2 - 1})/\sqrt{-3} = t$, we have

$$(\sqrt[3]{ab^2})^3 = (\sqrt{-3} t + 1)^3, \text{ so } -3\sqrt{-3} (t^3 - t) = ab^2 - 1 + 9t^2 \text{ and at last we have the equation:}$$

$$t^6 + t^4 + \frac{(ab^2 - 1)^2}{27} + t^2 \cdot \frac{(1+2ab^2)}{3} = 0 \text{ which shows } t \text{ is an integral. We take}$$

$x = [1, t, \theta_0]$, then

$$\text{disc}(x) = \begin{vmatrix} 1 & \frac{\sqrt[3]{ab^2 - 1}}{\sqrt{-3}} & \frac{1 + \sqrt[3]{ab^2} + \sqrt[3]{a^2 b}}{3} \\ 1 & \rho \frac{\sqrt[3]{ab^2 - 1}}{\sqrt{-3}} & \frac{1 + \rho \sqrt[3]{ab^2} + \rho^2 \sqrt[3]{a^2 b}}{3} \\ 1 & \rho^2 \frac{\sqrt[3]{ab^2 - 1}}{\sqrt{-3}} & \frac{1 + \rho^2 \sqrt[3]{ab^2} + \rho \sqrt[3]{a^2 b}}{3} \end{vmatrix}^2$$

$$= \left[\frac{\theta_0}{3\sqrt{-3}} (\rho^2 \sqrt[3]{ab^2} - 1 - \rho \sqrt[3]{ab^2} + 1) + \frac{\theta_0^3}{3\sqrt{-3}} (\sqrt[3]{ab^2} - 1 - \rho^2 \sqrt[3]{ab^2} + 1) + \frac{\theta_0^6}{3\sqrt{-3}} (\rho \sqrt[3]{ab^2} - 1 - \sqrt[3]{ab^2} + 1) \right]^2$$

$\text{disc}(x) = \left[\frac{3ab(\rho^2 - \rho)}{3\sqrt{-3}} \right]^2 = a^2 b^2$. Since $d_{K_6/K_3} = f_0^2 = (ab)^2 = \text{disc}(x)$, then

$x = \left[1, \frac{\sqrt[3]{ab^2 - 1}}{\sqrt{-3}}, \frac{\sqrt[3]{ab^2} + \sqrt[3]{a^2 b} + 1}{3} \right]$ is a relative integral basis of O_{K_6}/O_{K_2} .

ILLUSTRATION 5.7. For $k_3 = Q(\sqrt[3]{10})$, $a \equiv \pm b \pmod{9}$, so

$$O_{K_6} = \left[1, \frac{\sqrt[3]{10} - 1}{\sqrt{-3}}, \frac{\sqrt[3]{10} + \sqrt[3]{10^2} + 1}{3} \right] \cdot O_{K_2}.$$

Here we will give another theorem for computing a relative integral basis of O_{K_6} over O_{K_2} for $\pm n = 3t+1$ no matter whether 3_1 is a P.I. in O_{K_3} or not.

THEOREM 5.8. Let $n = 3t+1$, $m = -n$ be square-free in $k_3 = Q(\sqrt[3]{n})$ for Type I, then

$$O_{K_6} = \left[\frac{1 + \sqrt[3]{n} + \sqrt[3]{n^2}}{\sqrt{-3}}, \sqrt[3]{n}, \sqrt[3]{n^2} \right] \cdot O_{K_2},$$

or

$$O_{K_6} = \left[\frac{1 - \sqrt[3]{n} + \sqrt[3]{n^2}}{\sqrt{-3}}, \sqrt[3]{n}, \sqrt[3]{n^2} \right] \cdot O_{K_2}.$$

PROOF. At first we will show that $t = (1 + \sqrt[3]{n} + \sqrt[3]{n^2})/\sqrt{-3}$ is an integral.

We take

$$t = \frac{(1 - \sqrt[3]{n})(1 + \sqrt[3]{n} + \sqrt[3]{n^2})}{(1 + \sqrt[3]{n}) \cdot \sqrt{-3}}, \quad \frac{(1-n) - t\sqrt{-3}}{-\sqrt{-3}t} = \frac{3}{\sqrt[3]{n}}, \quad \frac{((1-n) - t\sqrt{-3})^3}{3t^3 \cdot \sqrt{-3}} = n,$$

$$(1-n)^3 + 3t\sqrt{-3} - 3 + \sqrt{-3}(1-n)^2 - 9t^2(1-n) - nt^3 \cdot 3\sqrt{-3} = 0,$$

$$\left[\sqrt{-3}(3t^3 - 3t(1-n)^2 - 3nt^3) \right]^2 = \left[-(1-n)^3 + 9t^2(1-n) \right]^2,$$

or briefly:

$$-27(1-n)^2 t^6 - 27t^4(1-n)^2(2n+1) - 9t^2(1-n)^4 - (1-n)^6 = 0,$$

$$t^6 + (2n+1)t^4 + \frac{(1-n)^2}{3} \cdot t^2 + \frac{(1-n)^4}{27} = 0,$$

which shows that t is an integral. Now we take

$$x = \left[\frac{1 + \sqrt[3]{n} + \sqrt[3]{n^2}}{\sqrt{-3}}, \sqrt[3]{n}, \sqrt[3]{n^2} \right],$$

$$\text{disc}(x) = \begin{vmatrix} \frac{1 + \sqrt[3]{n} + \sqrt[3]{n^2}}{\sqrt{-3}} & \sqrt[3]{n} & \sqrt[3]{n^2} \\ \frac{1 + \rho \sqrt[3]{n} + \rho^2 \sqrt[3]{n^2}}{\sqrt{-3}} & \rho \sqrt[3]{n} & \rho^2 \sqrt[3]{n^2} \\ \frac{1 + \rho^2 \sqrt[3]{n} + \rho \sqrt[3]{n^2}}{\sqrt{-3}} & \rho^2 \sqrt[3]{n} & \rho \sqrt[3]{n^2} \end{vmatrix}^2$$

$$= n^2 \left[\frac{3}{\sqrt{-3}} \cdot (\rho^2 - \rho) + \frac{1 + \rho \sqrt[3]{n} + \rho^2 \sqrt[3]{n^2}}{\sqrt{-3}} \cdot (\rho^2 - \rho) + \frac{1 + \rho^2 \sqrt[3]{n} + \rho \sqrt[3]{n^2}}{\sqrt{-3}} \cdot (\rho^2 - \rho) \right]^2$$

$$= n^2 \left[\frac{\rho^2 - \rho}{\sqrt{-3}} (3 + 0 + 0) \right]^2 = 3^2 \cdot n^2$$

Since $d_{6/2} = f_0^2 = (3n)^2 = \text{disc}(x)$, then

$$O_{K_6} = \left[\frac{1 + \sqrt[3]{n} + \sqrt[3]{n^2}}{\sqrt{-3}}, \sqrt[3]{n}, \sqrt[3]{n^2} \right] \cdot O_{K_2}.$$

We can apply the same proof for $m = -n$.

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