

EXTENSIONS OF THE HEISENBERG-WEYL INEQUALITY

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ABSTRACT. In this paper a number of generalizations of the classical Heisenberg-Weyl uncertainty inequality are given. We prove the n-dimensional Hirschman entropy inequality (Theorem 2.1) from the optimal form of the Hausdorff-Young theorem and deduce a higher dimensional uncertainty inequality (Theorem 2.2). From a general weighted form of the Hausdorff-Young theorem, a one-dimensional weighted entropy inequality is proved and some weighted forms of the Heisenberg-Weyl inequalities are given.

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1. INTRODUCTION.

Let \hat{f} be the Fourier transform of f defined by

$$\hat{f}(x) = \int e^{-2\pi ixy} f(y) dy, \quad x \in \mathbb{R}.$$

If $f \in L^2(\mathbb{R})$ with L^2 -norm $\|f\|_2 = 1$, then by Plancherel's theorem $\|\hat{f}\|_2 = 1$, so that $|f(x)|^2$ and $|\hat{f}(y)|^2$ are probability frequency functions. The variance of a probability frequency function g is defined by

$$V[g] = \int_{\mathbb{R}} (x-m)^2 g(x) dx \quad \text{where} \quad m = \int_{\mathbb{R}} xg(x) dx$$

is the mean. With these notations, the Heisenberg uncertainty principle of quantum mechanics can be stated in terms of the Fourier transform by the inequality

$$V[|f|^2]V[|\hat{f}|^2] \geq (16\pi^2)^{-1}. \tag{1.1}$$

In the sequel, we assume without loss of generality that the mean $m = 0$. If g is a probability frequency function, then the entropy of g is defined by

$$E[g] = \int_{\mathbb{R}} g(x) \log g(x) dx.$$

With f as above, Hirschman [1] proved that

$$E[|f|^2] + E[|\hat{f}|^2] \leq E_H \tag{1.2}$$

with $E_H = 0$, and suggested that (1.2) holds with $E_H = \log 2 - 1$. If E_H has that form, then by an inequality of Shannon and Weaver [2] it follows that (1.2) implies (1.1).

Using the Babenko-Beckner optimal form of the Hausdorff-Young inequality ([3])

$$\|\hat{f}\|_{p'} \leq A(p) \|f\|_p, \quad 1 < p < 2, \quad A(p) = [p^{1/p}(p')^{-1/p}]^{1/2}, \tag{1.3}$$

in Hirschman's proof of (1.2), then as Beckner [4] noted, (1.2) holds with $E_H = \log 2 - 1$.

A modest extension of (1.1) is obtained as follows: Let f on \mathbb{R} be differentiable, such that $f(0) = 0$. Then Hölder's and Hardy's inequality [4, Theorem 3.27] yield with $1 < p \leq 2$

$$\begin{aligned} \int_0^\infty |f(x)|^2 dx &\leq \left(\int_0^\infty |x f(x)|^p dx\right)^{1/p} \left(\int_0^\infty |f(x)/x|^{p'} dx\right)^{1/p'} \\ &\leq p \left(\int_0^\infty |x f(x)|^p dx\right)^{1/p} \left(\int_0^\infty |f'(x)|^{p'} dx\right)^{1/p'}. \end{aligned}$$

Applying this estimate also to $f(-x)$, then

$$\begin{aligned} \|f\|_2^2 &= \int_0^\infty |f(x)|^2 dx + \int_0^\infty |f(-x)|^2 dx \\ &\leq p \left[\left(\int_0^\infty |x f(x)|^p dx\right)^{1/p} \left(\int_0^\infty |f'(x)|^{p'} dx\right)^{1/p'} \right. \\ &\quad \left. + \left(\int_0^\infty |x f(-x)|^p dx\right)^{1/p} \left(\int_0^\infty |f'(-x)|^{p'} dx\right)^{1/p'} \right] \\ &\leq p \left(\int_{-\infty}^\infty |x f(x)|^p dx\right)^{1/p} \left(\int_{-\infty}^\infty |f'(x)|^{p'} dx\right)^{1/p'}, \end{aligned}$$

where the last inequality follows from Hölder's inequality. Now by (1.3) and the fact that $\hat{f}'(y) = 2\pi i y \hat{f}(y)$ we obtain

THEOREM 1.1. If $f \in S(\mathbb{R})$ and $f(0) = 0$, then for $1 < p \leq 2$

$$\|f\|_2^2 \leq 2\pi p A(p) \|x f\|_p \|y \hat{f}\|_p. \tag{1.4}$$

Note that the constant in (1.4) is slightly better than that in [4, §1.4] but unlikely best possible.

The purpose of this paper is to give extensions of the Heisenberg-Weyl inequality (1.1). In the next section a new proof of the entropy inequality (1.2) for functions on \mathbb{R}^n is given and an n -dimensional Heisenberg-Weyl inequality is deduced. The n -dimensional generalization of inequality (1.4) is also given in the next section. The two inequalities are quite different, even in the case $p = 2$, but depend strongly on the sharp Hausdorff-Young inequality. In the third section a weighted form of the Heisenberg-Weyl inequality in one dimension is obtained from a weighted form of the Hausdorff-Young inequality ([5][6][7][8]). Unlike the constant $A(p)$ in (1.3) the constant of the weighted Hausdorff-Young inequality (3.3) of (Theorem 3.1) is far from sharp. If the constant is not too large, then a weighted form of Hirschman's entropy inequality can also be given, from which another uncertainty inequality is deduced.

Throughout, $p' = p/(p-1)$, with $p' = \infty$ if $p = 1$, is the conjugate index of p , and similarly for other letters. $S(\mathbb{R}^n)$ is the Schwartz class of slowly increasing functions on \mathbb{R}^n . We say g is in the weighted L_w^r -space with weight w , if $wg \in L^r$ and norm $\|g\|_{r,w} = \|wg\|_r$. If $x \in \mathbb{R}^n$, then $x = (x_1, x_2, \dots, x_n)$ and $dx = dx_1 \dots dx_n$ the n -dimensional Lebesgue measure. $f_i(x)$, $x \in \mathbb{R}^n$ denotes the partial derivative of f with respect to the i^{th} component and $f_{ij} = (f_i)_j$. The letter C denotes a constant which may be different at different occurrences, but is independent of f .

2. THE HIRSCHMAN INEQUALITY.

The Fourier transform of f on \mathbb{R}^n is given by

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} f(y) dy, \quad x \in \mathbb{R}^n, \quad x \cdot y = x_1 y_1 + \dots + x_n y_n;$$

and the entropy of a function on \mathbb{R}^n is defined as before with \mathbb{R} replaced by \mathbb{R}^n . We shall need the following well known result (c.f. [9; §13.32 ii]):

If $\int_X d\mu = 1$, then

$$\lim_{p \rightarrow +\infty} \left(\int_X |f|^p d\mu \right)^{1/p} = \exp \int_X \log |f| d\mu. \quad (2.1)$$

Using this fact we obtain easily the n -dimensional form of Hirschman's inequality (1.2)

THEOREM 2.1. If $f \in L^2(\mathbb{R}^n)$ such that $\|f\|_2 = \|\hat{f}\|_2 = 1$, then

$$E[|f|^2] + E[|\hat{f}|^2] \leq n[\log 2 - 1], \quad (2.2)$$

whenever the left side has meaning.

PROOF. Let $f \in (L^1 \cap L^2)(\mathbb{R}^n)$, then $f \in L^p(\mathbb{R}^n)$, $1 < p < 2$, and by the n -dimensional form of the sharp Hausdorff-Young inequality [3] (that is, (1.3) with $A(p)$ replaced by $[A(p)]^n$) we obtain with $p = 2-r$, $r > 0$ and $p' = 2-r'$, $r' < 0$

$$\left(\int_{\mathbb{R}^n} |\hat{f}(y)|^{2-r'} dy \right)^{-1/r'} \leq \left[\frac{(2-r)^{1/2r}}{(2-r')^{-1/(2r')}} \right]^n \left(\int_{\mathbb{R}^n} |f(x)|^{2-r} dx \right)^{1/r}.$$

Now let $d\hat{\mu} = |\hat{f}(y)|^2 dy$ and $d\mu = |f(x)|^2 dx$, then $\int_{\mathbb{R}^n} d\hat{\mu} = \int_{\mathbb{R}^n} d\mu = 1$, so that the inequality becomes

$$\left(\int_{\mathbb{R}^n} |\hat{f}(y)|^{-r'} d\hat{\mu} \right)^{-1/r'} / \left(\int_{\mathbb{R}^n} (1/|f(x)|)^{r'} d\mu \right)^{1/r} \leq \left[\frac{(2-r)^{-1/(2r)}}{(2-r')^{-1/(2r')}} \right]^n.$$

But as $r \rightarrow 0+$, $-r' \rightarrow 0+$, so that by (2.1)

$$\begin{aligned} & \exp \left(\int_{\mathbb{R}^n} \log |\hat{f}(y)| d\hat{\mu} \right) / \exp \left(\int_{\mathbb{R}^n} \log (|f(x)|^{-1}) d\mu \right) \\ &= \exp \left(\int_{\mathbb{R}^n} |\hat{f}(y)|^2 \log |\hat{f}(y)| dy + \int_{\mathbb{R}^n} |f(x)|^2 \log |f(x)| dx \right) \leq \lim_{r \rightarrow 0} \frac{(2-r)^{n/(2r)}}{(2-r')^{-n/(2r')}} \\ &= 2^n / 2e^{-n/2}. \end{aligned}$$

Taking logarithms on both sides we get

$$\int_{\mathbb{R}^n} |\hat{f}(y)|^2 \log |\hat{f}(y)| dy + \int_{\mathbb{R}^n} |f(x)|^2 \log |f(x)| dx \leq \frac{n}{2} [\log 2 - 1]$$

and this implies (2.2) in the case $f \in (L^1 \cap L^2)(\mathbb{R}^n)$.

If $f \in L^2$ the result is obtained as in [1] only now one takes for ω_T , $\omega_\epsilon(x) = e^{-\pi\epsilon|x|^2}$ and for $\hat{\omega}_T$, $\hat{\omega}_\epsilon(y) = \epsilon^{-n/2} e^{-\pi|y|^2/\epsilon}$. We omit the details.

If $|g| \in L^2(\mathbb{R})$ is a probability frequency function, then the relation between entropy and variance is expressed by $E[|g|^2] \geq -\frac{1}{2} - \frac{1}{2} \log(2\pi V[|g|^2])$ ([2; p. 55-56]). The n -dimensional form of this inequality is given in the following lemma:

LEMMA 2.1. ([2; p. 56-57]). Let $g \in L^2(\mathbb{R}^n)$ with $\|g\|_2 = 1$. If $B = (b_{ij})$ is the matrix with entries

$$b_{ij} = V[|g|^2] = \int_{\mathbb{R}^n} x_i x_j |g(x)|^2 dx, \quad i, j = 1, 2, \dots, n;$$

then

$$E[|g|^2] \geq \frac{n}{2} \log(2\pi |b_{ij}|^{1/n}) - n/2$$

where $|b_{ij}| = \det B$.

Using the lemma and Theorem 2.1, we easily establish an n -dimensional extension of the Heisenberg-Weyl inequality.

THEOREM 2.2. Let $f \in L^2(\mathbb{R}^n)$ with $\|f\|_2 = \|\hat{f}\|_2 = 1$ and

$$b_{ij} = \int_{\mathbb{R}^n} x_i x_j |f(x)|^2 dx, \quad \hat{b}_{ij} = \int_{\mathbb{R}^n} y_i y_j |\hat{f}(y)|^2 dy,$$

$i, j = 1, 2, \dots, n$; be the entries of the matrices B and \hat{B} respectively, then

$$(\det B)(\det \hat{B}) \geq (16 \pi^2)^{-n}.$$

PROOF. By (2.2) and Lemma 2.1,

$$\begin{aligned} n[\log 2 - 1] &\geq E[|f|^2] + E[|\hat{f}|^2] \\ &\geq -\frac{n}{2} \log(2\pi |b_{ij}|^{1/n}) - \frac{n}{2} \log(2\pi |\hat{b}_{ij}|^{1/n}) - n, \end{aligned}$$

so that

$$\log 2 \geq -\frac{1}{2} \log(4\pi^2 |b_{ij}|^{1/n} |\hat{b}_{ij}|^{1/n}).$$

But then

$$4 \geq 1/[(\det B)^{1/n} (\det \hat{B})^{1/n} 4\pi^2],$$

which implies the result.

Clearly, if $n = 1$ we obtain at once (1.1). If $n = 2$ then

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}^2} x_1^2 |f|^2 dx, & \int_{\mathbb{R}^2} x_1 x_2 |f|^2 dx \\ \int_{\mathbb{R}^2} x_2 x_1 |f|^2 dx, & \int_{\mathbb{R}^2} x_2^2 |f|^2 dx \end{pmatrix},$$

with a similar expression for \hat{B} . Applying Theorem 2.2 we obtain

$$\begin{aligned} (\det B)(\det \hat{B}) &= [(\int_{\mathbb{R}^2} x_1^2 |f|^2 dx)(\int_{\mathbb{R}^2} x_2^2 |f|^2 dx) - (\int_{\mathbb{R}^2} x_1 x_2 |f|^2 dx)^2] \\ &\quad \cdot [(\int_{\mathbb{R}^2} y_1^2 |\hat{f}|^2 dy) - (\int_{\mathbb{R}^2} y_1 y_2 |\hat{f}|^2 dy)^2] \geq (16 \pi^2)^{-2}. \end{aligned}$$

If we denote the bracketed terms above by $D[|f|^2]$ and $D[|\hat{f}|^2]$, the discrepancy of Schwarz's inequality, or the difference between variance and covariance of $|f|^2$ and $|\hat{f}|^2$, then the two dimensional Heisenberg-Weyl inequality shows that the discrepancies of $|f|^2$ and $|\hat{f}|^2$ cannot both be small; $D[|f|^2] D[|\hat{f}|^2] \geq (16 \pi^2)^{-2}$.

A different generalization of (1.1) may be obtained along the lines of Theorem 1.1.

THEOREM 2.3. Let $f \in S(\mathbb{R}^n)$, such that $f(x_1, x_2, \dots, x_n) = 0$, whenever $x_i = 0$ for some i . If $1 < p \leq 2$ and $A(p)$ is the constant of (1.3), then

$$\|f\|_2^2 \leq [2\pi p A(p)]^n \|x_1 \dots x_n f\|_p \|y_1 \dots y_n \hat{f}\|_p.$$

PROOF. We only give the proof for $n = 2$ since the general case follows in exactly the same way. Let $f_{21}(x, y) = g(x, y)$, then

$$f(x, y) = \int_0^x \int_0^y g(s, t) dt ds$$

and by Hölder's and the two dimensional Hardy inequality, with $\mathbb{R}_+^2 = (0, \infty) \times (0, \infty)$,

$$\begin{aligned} \int_{\mathbb{R}_+^2} |f(x, y)|^2 dx dy &\leq (\int_{\mathbb{R}_+^2} |xy f(x, y)|^p dx dy)^{1/p} (\int_{\mathbb{R}_+^2} |f(x, y)/xy|^p dx dy)^{1/p'} \\ &\leq p^2 (\int_{\mathbb{R}_+^2} |xy f(x, y)|^p dx dy)^{1/p} (\int_{\mathbb{R}_+^2} |f_{21}(x, y)|^p dx dy)^{1/p'}. \end{aligned}$$

On applying this estimate four times we obtain with $d\mu = dx dy$

$$\|f\|_2^2 = \int_{\mathbb{R}_+^2} (|f(x, y)|^2 + |f(x, -y)|^2 + |f(-x, -y)|^2 + |f(-x, y)|^2) d\mu$$

$$\begin{aligned}
&\leq p^2 \left\{ \left(\int_{\mathbb{R}_+^2} |xy f(x,y)|^p d\mu \right)^{1/p} \left(\int_{\mathbb{R}_+^2} |f_{21}(x,y)|^{p'} d\mu \right)^{1/p'} \right. \\
&\quad + \left(\int_{\mathbb{R}_+^2} |xy f(x,-y)|^p d\mu \right)^{1/p} \left(\int_{\mathbb{R}_+^2} |f_{21}(x,-y)|^{p'} d\mu \right)^{1/p'} \\
&\quad + \left(\int_{\mathbb{R}_+^2} |xy f(-x,-y)|^p d\mu \right)^{1/p} \left(\int_{\mathbb{R}_+^2} |f_{21}(-x,-y)|^{p'} d\mu \right)^{1/p'} \\
&\quad \left. + \left(\int_{\mathbb{R}_+^2} |xy f(-x,y)|^p d\mu \right)^{1/p} \left(\int_{\mathbb{R}_+^2} |f_{21}(-x,y)|^{p'} d\mu \right)^{1/p'} \right\} \\
&\leq p^2 \left\{ \left(\int_{\mathbb{R}_+^2} |xy|^p [|f(x,y)|^p + |f(x,-y)|^p + |f(-x,-y)|^p + |f(-x,y)|^p] d\mu \right)^{1/p} \right. \\
&\quad \times \left(\int_{\mathbb{R}_+^2} |f_{21}(x,y)|^{p'} + |f_{21}(x,-y)|^{p'} \right. \\
&\quad \quad \left. + |f_{21}(-x,-y)|^{p'} + |f_{21}(-x,y)|^{p'} d\mu \right)^{1/p'} \left. \right\} \\
&= p^2 \left(\int_{\mathbb{R}_+^2} |xy f(x,y)|^p d\mu \right)^{1/p} \left(\int_{\mathbb{R}_+^2} |f_{21}(x,y)|^{p'} d\mu \right)^{1/p'},
\end{aligned}$$

where the last inequality follows from Hölder's inequality. But by the sharp form of the Hausdorff-Young inequality with $n = 2$ we obtain $\|f\|_2^2 \leq [p A(p)]^2 \|xyf\|_p \|\hat{f}_{21}\|_{p'}$. Since $(\hat{f}_{-1})(s,t) = 4\pi^2 st \hat{f}(s,t)$ the result follows.

3. WEIGHTED HIRSCHMAN ENTROPY INEQUALITY AND WEIGHTED HEISENBERG-WEYL INEQUALITY.

The results of the last section show that the Heisenberg-Weyl inequality is a consequence of the Hausdorff-Young theorem. Recently a number of weighted Hausdorff-Young inequalities have been obtained [5], [6], [7] and [8]. We shall use these results in this section to obtain a weighted Hirschman entropy inequality as well as weighted form of the Heisenberg-Weyl inequality. Here we consider weighted extensions in \mathbb{R}^1 only.

Recall that if g is a Lebesgue measurable function on \mathbb{R} , then the equi-measurable decreasing rearrangement of g is defined by $g^*(t) = \inf\{y > 0: |\{x \in \mathbb{R}: |g(x)| > y\}| \leq t\}$, where $y > 0$ and $|E|$ denotes Lebesgue measure of the set E . Clearly, if g is an even function on \mathbb{R} , decreasing on $(0, \infty)$, then for $t > 0$, $g^*(t) = g(t/2)$. We shall use this fact below.

DEFINITION 3.1. Let u and v be locally integrable functions of \mathbb{R} . We write $(u,v) \in F_{p,q}^*$, $1 \leq p \leq q < \infty$, if

$$\sup \left(\int_0^s [u^*(t)]^q dt \right)^{1/q} \left(\int_0^{1/s} [(1/v)^*(t)]^{p'} dt \right)^{1/p'} < \infty, \quad (3.1)$$

where in the case $p = 1$ the second integral is replaced by the essential supremum of $(1/v)^*(t)$ over $(0, 1/s)$.

If u and $1/v$ are even and decreasing on $(0, \infty)$ then (3.1) is equivalent to

$$\sup_{s>0} \left(\int_0^{s/2} [u(x)]^q dx \right)^{1/q} \left(\int_0^{1/(2s)} v(x)^{-p'} dx \right)^{1/p'} < \infty \quad (3.2)$$

and in this case we write $(u, v) \in F_{p,q}$.

The weighted Hausdorff-Young inequality is given in the following theorem:

THEOREM 3.1. ([5; Theorem 1.1]). Suppose $(u,v) \in F_{p,q}^*$, $1 \leq p \leq q < \infty$ and $f \in L_v^p$.

(i) If $\lim_{n \rightarrow \infty} \|f_n - f\|_{p,v} = 0$ for a sequence of simple functions, then $\{\hat{f}_n\}$ con-

verges in L_u^q to a function $\hat{f} \in L_u^q$. \hat{f} is independent of the sequence $\{\hat{f}_n\}$ and is called

the Fourier transform of f .

(ii) there is a constant $B > 0$ such that for all $f \in L^p_{v^p}$

$$\|\hat{f}\|_{q,u} < B \|f\|_{p,v} \tag{3.3}$$

(iii) If $g \in L^q_{1/u}$, $q > 1$, then Parseval's formula

$$\int_{\mathbb{R}} \hat{f}(y)g(y)dy = \int_{\mathbb{R}} f(t)\hat{g}(t)dt$$

holds.

We note ([5], [6], [8]) that Theorem 3.1 is sharp in the sense that if u and v are even and satisfy (3.3), then (u, v) satisfies (3.2). The constant B in (3.3) is not sharp, however it is of the form $B = k.C$ where $k = k(p,q)$ is independent of u and v and C is the supremum of (3.1), and in the case $u, 1/v$ decreasing and even the supremum (3.2).

A special case of Theorem 3.1 is the following:

COROLLARY 3.1. Suppose $f \in L^{p/2}_v$, $(u^{1-2/p}, v^{1-2/p}) \in F^*_{p,p}$, $1 < p < 2$, where u and v are even, decreasing as $(0, \infty)$ then

$$\left(\int_{\mathbb{R}} u(y)^{p-2} |\hat{f}(y)|^p < k.C \left(\int_{\mathbb{R}} v(x)^{p-2} |f(x)|^p dx\right)^{1/p} \tag{3.4}$$

where $C_p = \sup_{s>0} \left(\int_0^{s/2} u(x)^{p-2} dx\right)^{1/p} \left(\int_0^{1/(2s)} v(x)^{(2-p)p/p} dx\right)^{1/p}$.

Utilizing the last result we now give a weighted form of Hirschman's entropy inequality.

PROPOSITION 3.1. Suppose $f \in L^2 \cap L^1_{1/v}$, where u and v satisfy the conditions of Corollary 3.1. If $\|f\|_2 = 1$ and (3.4) holds with $0 < k \leq 2$ and C_p remains bounded as $p \rightarrow 2$, then

$$\begin{aligned} \int_{\mathbb{R}} |\hat{f}(y)|^2 \log |u(y)\hat{f}(y)|^2 dy + \int_{\mathbb{R}} |f(x)|^2 \log |v(x)f(x)|^2 dx \\ \leq 2 \log k + 8 \sup_{s>0} \left(\int_0^{s/2} u(x)^{1/(2s)} dx\right) \log |u(x)v(y)| dx dy. \end{aligned}$$

PROOF. Since $f \in L^{p/2}_v$, $1 < p < 2$, we apply Corollary 3.1 with $p = 2-r$, $r > 0$, $p' = 2-r'$, $r' < 0$ and $d\mu(y) = |\hat{f}(y)|^2 dy$, $d\mu(x) = |f(x)|^2 dx$. Then (3.4) has the form

$$\begin{aligned} \left(\int_{\mathbb{R}} |u(y)\hat{f}(y)|^{-r'} d\mu\right)^{1/(2-r')} \leq k \sup_{s>0} \left[\int_0^{s/2} u(x)^{-r'} dx \int_0^{1/(2s)} v(x)^{-r'} dx\right]^{1/(2-r')} \\ \cdot \left(\int_{\mathbb{R}} |v(x)f(x)|^{-r'} d\mu\right)^{1/(2-r)} \end{aligned}$$

or, on raising the inequality to the power $(2-r')(-1/r')$, equivalently

$$\left(\int_{\mathbb{R}} |u(y)\hat{f}(y)|^{-r'}\right)^{-1/r'} / \left(\int_{\mathbb{R}} |v(x)f(x)|^{-r'} d\mu\right)^{1/r} \leq k \left(\frac{k}{2}\right)^{-2/r'} M_r,$$

where

$$M_r = \sup_{s>0} \left[\int_0^{s/2} u(x)^{1/(2s)} dx \int_0^{1/(2s)} v(y)^{-r'} dy\right]^{-1/r'}$$

Given $\epsilon > 0$ there is an $s_0 > 0$ such that

$$M_r \leq \left[\int_0^{s_0/2} u(x)^{1/(2s)} dx \int_0^{1/(2s)} v(y)^{-r'} dy\right]^{-1/r'} + \epsilon$$

so that

$$\begin{aligned} & \left(\int_{\mathbb{R}} |u(y)\hat{f}(y)|^{-r'} d\hat{\mu} \right)^{-1/r'} / \left(\int_{\mathbb{R}} |v(x)f(x)|^{-r} d\mu \right)^{1/r} \\ & \leq k \left(\int_0^{s_0/2} \int_0^{1/(2s_0)} [u(x)v(y)]^{-r'} dx dy \right)^{-1/r'} + \epsilon, \end{aligned} \tag{3.5}$$

where we used the fact that $k/2 \leq 1$. Now as $r \rightarrow 0+$, $r' \rightarrow 0-$, then on applying (2.1) to both sides of (3.5) we obtain

$$\begin{aligned} & \exp\left(\int_{\mathbb{R}} \log|u(y)\hat{f}(y)| d\hat{\mu}\right) / \exp\left(\int_{\mathbb{R}} \log|1/[v(x)f(x)]| d\mu\right) \\ & \leq k \left[\exp \int_0^{s_0/2} \int_0^{1/(2s_0)} \log[v(y)u(x)] dx dy + \epsilon \right] \\ & \leq k \left[\exp \sup_{s>0} \int_0^{s/2} \int_0^{1/(2s)} \log[u(x)v(y)] dx dy + \epsilon \right]. \end{aligned}$$

But $\epsilon > 0$ is arbitrary so that on taking logarithms we have

$$\begin{aligned} & \frac{1}{2} \int_{-\infty}^{\infty} |\hat{f}(y)|^2 \log|u(y)\hat{f}(y)|^2 dy + \frac{1}{2} \int_{-\infty}^{\infty} |f(x)|^2 \log|v(x)f(x)|^2 dx \\ & \leq \log k + 4 \sup_{s>0} \left(\int_0^{s/2} \log u(x) dx + \int_0^{1/(2s)} \log v(y) dy \right) \end{aligned}$$

which yields the result.

Note that if $u = v \equiv 1$ and if $k \leq \sqrt{2/e}$ we obtain (2.2) with $n = 1$.

We can write the conclusion of Proposition 3.1 in the form

$$\begin{aligned} E[|f|^2] + E[|\hat{f}|^2] & \leq 2 \log k + 8 \sup_{s>0} \left(\int_0^{s/2} \int_0^{1/(2s)} \log|u(x)v(y)| dx dy \right) \\ & \quad - \int_{\mathbb{R}} |\hat{f}(y)|^2 \log|u(y)|^2 dy - \int_{\mathbb{R}} |f(x)|^2 \log|v(x)|^2 dx. \end{aligned}$$

But since $([2]) E[|f|^2] \geq -\frac{1}{2} - \frac{1}{2} \log(2\pi V[|f|^2])$ and also with f replaced by \hat{f} we obtain another uncertainty inequality

$$\begin{aligned} V[|f|^2] V[|\hat{f}|^2] & \geq \frac{k^{-4}}{4\pi^2 e^2} \exp[-16 \sup_{s>0} \int_0^{s/2} \int_0^{1/(2s)} \log|uv| dx dy] \\ & \quad \times \exp\left(2 \int_{\mathbb{R}} |\hat{f}|^2 \log|u|^2 dy\right) \exp\left(2 \int_{\mathbb{R}} |f|^2 \log|v|^2 dx\right). \end{aligned}$$

If $u = v \equiv 1$ and $k = \sqrt{2/e}$ in this estimate we obtain (1.1).

THEOREM 3.2. (Heisenberg-Weyl inequality). If $(1/u, v) \in F_{p,q}^*$, $1 \leq p \leq q < \infty$ and $f \in S(\mathbb{R})$, then

$$\| |f| \|_2^2 \leq C \left(\int_{\mathbb{R}} |xu(x)f(x)|^{q'} dx \right)^{1/q'} \left(\int_{\mathbb{R}} |v(y)y\hat{f}(y)|^p dy \right)^{1/p}. \tag{3.6}$$

PROOF. Integration by parts and Hölder's inequality show that for $1 \leq q < \infty$

$$\begin{aligned} \| |f| \|_2^2 & \leq 2 \int_{\mathbb{R}} |x| |f(x)| |f'(x)| dx \\ & \leq 2 \left(\int_{\mathbb{R}} |xu(x)f(x)|^{q'} dx \right)^{1/q'} \left(\int_{\mathbb{R}} |f'(x)/u(x)|^q dx \right)^{1/q} \\ & \leq 2C \left(\int_{\mathbb{R}} |xu(x)f(x)|^{q'} dx \right)^{1/q'} \left(\int_{\mathbb{R}} |v(y)\hat{f}'(y)|^p dx \right)^{1/p}, \end{aligned}$$

where the last inequality follows from (3.3). Since $\hat{f}'(y) = 2\pi iy \hat{f}(y)$ the result follows.

Note that the case $p = 1$ also holds, provided the second integral in the $F_{p,q}^*$ condition is interpreted as the essential supremum of $(1/v)^*$ over $(0, 1/s)$.

The same result holds also if we take $(1/u, v) \in F_{p,q}$.

Observe also that the case $u = v \equiv 1$ and $q = p'$, $1 < p \leq 2$ reduces to (1.4), but with a different constant.

Weighted inequalities of the form (3.6) were also obtained by Cowling and Price [3] but by quite different methods.

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Special Issue on Decision Support for Intermodal Transport

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