

RESEARCH NOTES

HOMOMORPHISMS OF COMPLETE n -PARTITE GRAPHS

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ABSTRACT. It is shown that for every homomorphism ϕ of a graph G there exists a contraction θ_ϕ on \bar{G} , the complement of G , such that $\overline{\phi(G)} = \theta_\phi(\bar{G})$ if and only if G is a complete n -partite graph.

KEY WORDS AND PHRASES. *Homomorphisms of graphs, contractions of graphs, complete n -partite graph.*

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By a graph G we mean a set $V(G)$ of vertices together with a set $E(G)$ of unordered pairs of distinct vertices in $V(G)$, called edges. An elementary homomorphism of a graph G is the identification of two non-adjacent vertices of G , and a homomorphism is a sequence of elementary homomorphisms. Thus a homomorphism of a graph G onto a graph H is a function ϕ from $V(G)$ onto $V(H)$ such that whenever u and v are adjacent in G , $\phi(u)$ and $\phi(v)$ are adjacent in H . Likewise, an elementary contraction of a graph G is the identification of two adjacent vertices of G , and a contraction is a sequence of elementary contractions. Thus a contraction of a graph G onto a graph H is a function θ from $V(G)$ onto $V(H)$ such that, for every $u \in V(H)$, $\theta^{-1}(u)$ is a connected subgraph of G and for every $uv \in E(H)$, there is at least one edge in G joining a vertex of $\theta^{-1}(u)$ with one of $\theta^{-1}(v)$. Now for every homomorphism ϕ of G there is a contraction θ_ϕ of \bar{G} , the complement of G , that we construct as follows: ϕ is a sequence of elementary homomorphisms $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ each of which identifies two non-adjacent vertices in G , so we let θ_ϕ be the sequence of elementary contractions $\theta_1, \theta_2, \dots, \theta_n$ such that θ_i identifies the same vertices in \bar{G} that ϵ_i identifies in G .

For $n \leq 1$, G is a complete n -partite graph if it is possible to partition $V(G)$

into n subsets V_1, V_2, \dots, V_n , called partite sets, such that if $u, v \in V_i$ then $uv \notin E(G)$, but if $u_i \in V_i$ and $u_j \in V_j$ where $i \neq j$, then $u_i u_j \in E(G)$. If $n = 1$, G is the null graph \overline{K}_p where p is the number of vertices in $V(G)$. If each V_i has exactly one vertex in it then G is the complete graph K_n . We also note that G is a complete n -partite graph if and only if \overline{G} is a disjoint union of n complete graphs. MAIN THEOREM. $\overline{\phi(G)} = \theta_\phi(\overline{G})$ for every homomorphism ϕ of G if and only if G is a complete n -partite graph.

Before proving this theorem we will need some additional groundwork. We note that any graph G is either connected or a union of disjoint connected graphs called components. Let $c(G)$ denote the number of components of the graph G . $\beta_0(G)$ will denote the point independence number of G , that is, the cardinality of the largest set of non-adjacent vertices in G . Clearly for any graph G , $\beta_0(G) \geq c(G)$. As usual $\chi(G)$ will denote the chromatic number of G , and Hedetniemi [2, p. 24] shows that for any graph G , $\chi(G) \geq \beta_0(\overline{G})$. Thus we have

$$(*) \quad \chi(G) \geq c(\overline{G})$$

for any graph G .

LEMMA. $\chi(G) = c(\overline{G})$ if and only if G is a complete n -partite graph.

PROOF. If G is a complete n -partite graph, then $\chi(G) = n$ and $c(\overline{G}) = n$.

On the other hand, suppose G is not a complete n -partite graph but $\chi(G) = c(\overline{G})$. If \overline{G} is connected then $c(\overline{G}) = 1$. However G must contain at least one edge (else G is a complete 1-partite graph) and so $\chi(G) \geq 2$, and we are done. So we suppose $c(\overline{G}) = n > 1$. Thus \overline{G} is the union of n components at least one of which is not a complete graph. We add edges to each component of \overline{G} , and delete the corresponding edges of G , until each component of \overline{G} becomes a complete graph, and hence G becomes a complete n -partite graph, call it G' . Now $\chi(G') = n$, and Culik [1] has shown that adding exactly one edge to a complete n -partite graph increases the chromatic number by one. However, deleting edges from any graph will either leave the chromatic number unchanged or decrease it. Therefore, if e is any edge that was deleted from G to obtain G' we have

$$n + 1 = \chi(G' + e) \leq \chi(G) = n,$$

a contradiction, and the lemma is proved.

PROOF OF THE MAIN THEOREM. Suppose G is a complete n -partite graph with partite sets V_1, V_2, \dots, V_n . Clearly the identity homomorphism and its related contraction, the null contraction, satisfy the given equation. Let ϵ be any elementary homomorphism of G . Then ϵ will identify two non-adjacent vertices and so will be restricted to some V_i . In fact $\epsilon(G)$ will still be a complete n -partite graph with one less vertex in V_i . Likewise the related elementary contraction θ_ϵ will be restricted to the component of \overline{G} that corresponds to V_i in G .

However every component of G is a complete graph and any elementary contraction on a complete graph yields a complete graph on one less vertex. Thus $\overline{\varepsilon(G)} = \theta_c(\overline{G})$ for every elementary homomorphism of G , and so $\overline{\phi(G)} = \theta_\phi(\overline{G})$ for every homomorphism ϕ of G .

Now suppose G is not a complete n -partite graph but $\overline{\phi(G)} = \theta_\phi(\overline{G})$ for every homomorphism ϕ of G . Let $\nu(G) = m$, then we know [2, p. 10] that the homomorphic image of G with the smallest number of vertices is K_m . On the other hand \overline{G} is the union of $\ell \geq 1$ components, and so the image of \overline{G} with the smallest number of vertices under a contraction will be $\overline{K_\ell}$. Since G is not a complete n -partite graph, from the Lemma and (*) we have $m > \ell$. Therefore \overline{G} is not contractable to $\overline{K_m}$, since any image of G under a contraction that has m vertices must have enough edges to be further contractable to $\overline{K_\ell}$. Hence for any homomorphism ϕ of G onto K_m we have $\overline{\phi(G)} \neq \theta_\phi(\overline{G})$ and we are done.

From the method used in the above proof it is evident that we also have the following:

THEOREM. There is a one-to-one correspondence between homomorphisms of G and the contractions of \overline{G} if and only if G is a complete n -partite graph. Otherwise the number of homomorphisms of G will be less than the number of contractions of \overline{G} .

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