

## A NOTE ON SUBORDINATION

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**ABSTRACT.** Suffridge showed a result for subordinate functions. The object of the present paper is to show some subordinate theorems with the aid of the result by Suffridge.

**KEY WORDS AND PHRASES.** *Subordinate, Starlike of order  $\alpha$ , convex of order  $\alpha$ .*

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### 1. INTRODUCTION.

Let  $f(z)$  and  $g(z)$  be analytic in the unit disk  $U = \{z: |z| < 1\}$ . A function  $f(z)$  is said to be subordinate to  $g(z)$  if there exists a function  $\phi(z)$  analytic in the unit disk  $U$  satisfying  $\phi(0) = 0$  and  $|\phi(z)| < 1$  ( $z \in U$ ) such that  $f(z) = g(\phi(z))$  for  $z \in U$ . We denote by  $f(z) \prec g(z)$  this relation. In particular, if  $g(z)$  is univalent in the unit disk  $U$  the subordination is equivalent to  $f(0) = g(0)$  and  $\text{range } f(z) \subset \text{range } g(z)$ .

This concept of subordination can be traced to Lindelöf [1], but Littlewood [2],[3] and Rogosinski [4], [5] introduced the term and discovered the basic properties. Recently Suffridge [6] and Hallenbeck and Ruscheweyh [7] studied the subordinate functions and showed many interesting results for subordinations.

Let  $A$  denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the unit disk  $U$ . Further let  $S$  be the subclass of  $A$  consisting of analytic and univalent functions in the unit disk  $U$ . Then a function  $f(z)$  of  $S$  is said to be starlike of order  $\alpha$  if and only if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U)$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ). We denote by  $S^*(\alpha)$  the class of all starlike functions of order  $\alpha$ . Further a function  $f(z)$  of  $S$  is said to be convex of order  $\alpha$  if and only if

$$\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U)$$

for some  $\alpha$  ( $0 \leq \alpha < 1$ ). And we denote by  $K(\alpha)$  the class of convex functions of

order  $\alpha$ . it is well-known that  $f(z) \in K(\alpha)$  if and only if  $zf'(z) \in S^*(\alpha)$ ,  $S^*(\alpha) \subset S^*$ ,  $K(\alpha) \subset K$ , and  $S^*(0) \equiv S^*$ ,  $K(0) \equiv K$  for  $\alpha = 0$ .

The classes  $S^*(\alpha)$  and  $K(\alpha)$  were first introduced by Robertson [8], and latter studied by Schild [9], MacGregor [10] and Pinchuk [11]. Further, recently, some] classes defined by using the extremal function  $z/(1-z)^{2(1-\alpha)}$  for  $S^*(\alpha)$  were studied by Ruscheweyh [12], Sheil-Small, Silverman and Silvia [13], Silverman and Silvia [14], and Ahuja and Silverman [15].

Our main tool in this paper is the following result by Suffridge [15].

LEMMA. Let the function  $f(z) = \sum_{n=2}^{\infty} a_n z^n$  be analytic in the unit disk  $u$  and the function  $g(z)$  be in the class  $S^*$ . If  $f(z)$  is subordinate to  $g(z)$ , that is,  $f(z) \prec g(z)$ , then

$$\int_0^z \frac{f(t)}{t} dt \prec \int_0^z \frac{g(t)}{t} dt$$

for  $z \in u(r) = \{z: |z| \leq r, 0 \leq r < 1\}$ .

2. SUBORDINATION THEOREMS.

In this section, we show some subordination theorems with the aid of Lemma.

THEOREM 1. Let the function  $f(z)$  defined by (1.1) be in the class of  $K(\alpha)$ . Then  $f'(re^{i\theta})$  ( $0 \leq r < 1$ ) is contained in the image domain of the closed disk  $u(r)$  under the function  $e^{4(\alpha-1)/(1-z)}$ . Further it lies for  $r \neq 0$  on the boundary of of this image domain if and only if

$$f(z) = \int_0^z e^{4(1-\alpha)/(1-\epsilon t)} dt. \tag{2.1}$$

where  $|\epsilon| = 1$ .

PROOF. Since  $f(z)$  is in the class  $K(\alpha)$ ,  $f(z)$  satisfies that

$$\operatorname{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha - 1 \tag{z \in u}.$$

Note that  $zf''(z)/f'(z) = 2a_2z + \dots$  is analytic in the unit disk  $u$ , and that the function  $z/(1-z)^2$  is starlike with respect to the origin and  $\operatorname{Re} \{z/(1-z)^2\} > -1/4$ . Hence we have that

$$\frac{zf''(z)}{f'(z)} \prec \frac{4(1-\alpha)z}{(1-z)^2} \tag{z \in u}.$$

Consequently, by using Lemma, it follows that  $\log f'(re^{i\theta})$  is contained in the image domain of  $u(r)$  under the function  $4(\alpha-1)/(1-z)$ , where  $\log$  is understood to be that branch which vanishes at the point one. Thus we can see that  $f'(re^{i\theta})$  lies for  $r \neq 0$  on the boundary of the image domain of  $u(r)$  under  $e^{4(\alpha-1)/(1-z)}$ . Further  $f'(re^{i\theta})$  lies for  $r \neq 0$  on the boundary of the image domain  $u(r)$  under

$$e^{4(\alpha-1)/(1-z)} \text{ if and only if } \frac{zf''(z)}{f'(z)} = \frac{4(1-\alpha)\epsilon z}{(1-\epsilon z)^2} \tag{|\epsilon| = 1},$$

hence further,  $f(z)$  is the function of the form (2.1). This completes the proof of the theorem.

**THEOREM 2.** Let the function  $f(z)$  defined by (1.1) be in the class  $S^*(\alpha)$ . Then  $f(re^{i\theta})/re^{i\theta}$  ( $0 \leq r < 1$ ) is contained in the image domain of the closed disk  $U(r)$  under the function  $e^{4(1-\alpha)/(1-z)}$ . Further it lies for  $r \neq 0$  on the boundary of this image domain if and only if  $f(z) = ze^{4(1-\alpha)/(1-\varepsilon z)}$ , where  $|\varepsilon| = 1$ .

**PROOF.** Since  $f(z) \in S^*(\alpha)$ ,  $f(z)$  satisfies that

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} - 1 \right\} > \alpha - 1 \quad (z \in U)$$

and the function  $zf'(z)/f(z) - 1 = a_2z + \dots$  is analytic in the unit disk  $U$ . Hence  $zf'(z)/f(z) - 1$  takes values in the image domain of the unit disk  $U$  under the function  $4(1-\alpha)z/(1-z)^2$ , that is,

$$\frac{zf'(z)}{f(z)} - 1 \prec \frac{4(1-\alpha)z}{(1-z)^2} \quad (z \in U).$$

By virtue of Lemma, we observe that  $\log f(re^{i\theta})/re^{i\theta}$  ( $0 \leq r < 1$ ) is contained in the image domain of  $U(r)$  under  $4(\alpha-1)/(1-z)$  and it lies for  $r \neq 0$  on the boundary of this image domain if and only if

$$\frac{zf'(z)}{f(z)} - 1 = \frac{4(1-\alpha)\varepsilon z}{(1-\varepsilon z)^2} \quad (|\varepsilon| = 1),$$

hence further,  $f(z) = ze^{4(1-\alpha)/(1-\varepsilon z)}$ . This gives the result we require.

Finally we show a theorem for functions  $f(z)$  satisfying  $\operatorname{Re}\{zf'(z)\} > \alpha$  ( $\alpha > 0$ ).

**THEOREM 3.** Let the function  $f(z)$  defined by (1.1) satisfy  $\operatorname{Re}\{zf'(z)\} > \alpha$  ( $\alpha > 0$ ). Then  $f(re^{i\theta})$  ( $0 \leq r < 1$ ) is contained in the image domain of the closed disk  $U(r)$  under the function  $-4\alpha/(1-z)$ . Further it lies for  $r \neq 0$  on the boundary of this domain if and only if  $f(z) = -4\alpha/(1-\varepsilon z)$ , where  $|\varepsilon| = 1$ .

**PROOF.** We note that the function  $zf'(z) = z + 2a_2z^2 + \dots$  takes values in the image domain of the unit disk  $U$  under the function  $-4\alpha z/(1-z)^2$  which belongs to the class  $S^*$ . Therefore we can prove the theorem by using the same technique as in the one of Theorem 1 with the aid of Lemma.

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