

ON A FIXED POINT THEOREM OF GREGUŠ

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(Received May 18, 1984)

ABSTRACT. We consider two selfmaps T and I of a closed convex subset C of a Banach space X which are weakly commuting in X , i.e.

$$\|T I x - I T x\| \leq a \|I x - T x\| \text{ for any } x \text{ in } X,$$

and satisfy the inequality

$$\|T x - T y\| \leq a \|I x - I y\| + (1 - a) \max \{ \|T x - I x\|, \|T y - I y\| \}$$

for all x, y in C , where $0 < a < 1$. It is proved that if I is linear and non-expansive in C and such that IC contains TC , then T and I have a unique common fixed point in C .

KEY WORDS AND PHRASES. *Common fixed point, Banach space.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 54H25, 47H10.

1. INTRODUCTION.

The second author [1], generalizing a result of Das and Naik [2], defined two mappings T and I of a metric space (X, d) into itself to be weakly commuting if

$$d(TIx, ITx) \leq d(Ix, Tx) \tag{1.1}$$

for all x in X . Two commuting mappings clearly satisfy (1.1) but the converse is not generally true as is shown with the following example:

EXAMPLE 1. Let $X = [0, 1]$ with the Euclidean metric and define T and I by

$$Tx = x/(x+4), \quad Ix = x/2$$

for all x in X . Then

$$\begin{aligned} d(TIx, ITx) &= \frac{x}{x+8} - \frac{x}{2x+8} = \frac{x^2}{2(x+8)(x+4)} \\ &\leq \frac{x^2 + 2x}{2(x+4)} = \frac{x}{2} - \frac{x}{x+4} = d(Ix, Tx) \end{aligned}$$

for all x in X but for any $x \neq 0$:

$$TIx = x/(x+8) > x/(2x+8) = ITx.$$

From now on, C denotes a closed convex subset of a Banach space X . In a recent paper Gregus [3] proved the following theorem:

THEOREM 1. Let T be a mapping of C into itself satisfying the inequality

$$\|Tx - Ty\| \leq a\|x - y\| + b\|Tx - x\| + c\|Ty - y\| \quad (1.2)$$

for all x, y in C , where $0 < a < 1$, $b \geq 0$, $c \geq 0$ and $a + b + c = 1$. Then T has a unique fixed point.

Mappings satisfying inequality (1.2) with $a = 1$ and $b = c = 0$ are called nonexpansive and were considered by Kirk [4].

Wong [5] studied mappings satisfying inequality (1.2) with $a = 0$ and $b = c = \frac{1}{2}$.

2. MAIN RESULTS.

We now prove the following generalization of Theorem 1:

THEOREM 2. Let T and I be two weakly commuting mappings of C into itself satisfying the inequality

$$\|Tx - Ty\| \leq a\|Ix - Iy\| + (1-a) \max \{ \|Tx - Ix\|, \|Ty - Iy\| \} \quad (2.1)$$

for all x, y in C , where $0 < a < 1$. If I is linear, nonexpansive in C and such that IC contains TC , then T and I have a unique common fixed point in C .

PROOF. Let $x = x_0$ be an arbitrary point in C and choose points x_1, x_2, x_3 in C such that

$$Ix_1 = Tx, Ix_2 = Tx_1, Ix_3 = Tx_2.$$

This can be done since IC contains TC . Then for $r = 1, 2, 3$ we have on using inequality (2.1)

$$\begin{aligned} \|Tx_r - Ix_r\| &= \|Tx_r - Tx_{r-1}\| \\ &\leq a\|Ix_r - Ix_{r-1}\| + (1-a) \max \{ \|Tx_r - Ix_r\|, \|Tx_{r-1} - Ix_{r-1}\| \} \\ &= a\|Tx_{r-1} - Ix_{r-1}\| + (1-a) \max \{ \|Tx_r - Ix_r\|, \|Tx_{r-1} - Ix_{r-1}\| \} \end{aligned}$$

and so

$$\|Tx_r - Ix_r\| \leq \|Tx_{r-1} - Ix_{r-1}\|.$$

It follows that

$$\|Tx_r - Ix_r\| \leq \|Tx - Ix\| \quad (2.2)$$

for $r = 1, 2, 3$.

Further

$$\begin{aligned} \|Tx_2 - Tx\| &\leq a\|Ix_2 - Ix\| + (1-a) \max \{ \|Tx_2 - Ix_2\|, \|Tx - Ix\| \} \\ &\leq a(\|Tx_1 - Ix_1\| + \|Tx - Ix\|) + (1-a) \|Tx - Ix\| \\ &\leq (1+a) \|Tx - Ix\| \end{aligned}$$

on using inequality (2.2). Thus

$$\|Tx_2 - Ix_1\| \leq (1+a) \|Tx - Ix\|. \quad (2.3)$$

We will now define a point z by

$$z = \frac{1}{2} x_2 + \frac{1}{2} x_3.$$

Since C is convex the point z is in C and being I linear, we have

$$Iz = \frac{1}{2} Ix_2 + \frac{1}{2} Ix_3 = \frac{1}{2} Tx_1 + \frac{1}{2} Tx_2.$$

It follows that

$$\begin{aligned} \|Tz - Iz\| &\leq \frac{1}{2} \|Tz - Tx_1\| + \frac{1}{2} \|Tz - Tx_2\| \\ &\leq \frac{1}{2} [a\|Iz - Ix_1\| + (1-a) \max \{\|Tz - Iz\|, \|Tx_1 - Ix_1\|\}] \\ &\quad + \frac{1}{2} [a\|Iz - Ix_2\| + (1-a) \max \{\|Tz - Iz\|, \|Tx_2 - Ix_2\|\}] \\ &= \frac{1}{2} a(\|Iz - Ix_1\| + \|Iz - Ix_2\|) + (1-a) \max \{\|Tz - Iz\|, \|Tx - Ix\|\} \end{aligned}$$

on using inequalities (2.1) and (2.2). Now

$$\begin{aligned} \|Iz - Ix_1\| &\leq \frac{1}{2} \|Ix_2 - Ix_1\| + \frac{1}{2} \|Ix_3 - Ix_1\| \\ &= \frac{1}{2} \|Tx_1 - Ix_1\| + \frac{1}{2} \|Tx_2 - Ix_1\| \\ &\leq (1 + \frac{1}{2} a) \|Tx - Ix\| \end{aligned}$$

from inequalities (2.2) and (2.3) and

$$\|Iz - Ix_2\| = \frac{1}{2} \|Ix_3 - Ix_2\| = \frac{1}{2} \|Tx_2 - Ix_2\| \leq \frac{1}{2} \|Tx - Ix\|.$$

It follows that

$$\|Tz - Iz\| \leq \frac{1}{4} a(3 + a) \|Tx - Ix\| + (1-a) \max \{\|Tz - Iz\|, \|Tx - Ix\|\}$$

and so

$$\|Tz - Iz\| \leq \lambda \|Tx - Ix\|$$

where

$$\lambda = (4 - a + a^2)/4 < 1.$$

We therefore have

$$\inf \{\|Tz - Iz\| : z = \frac{1}{2} x_2 + \frac{1}{2} x_3\} \leq \lambda \cdot \inf \{\|Tx - Ix\| : x \in C\}$$

and since we obviously have

$$\inf \{\|Tz - Iz\| : z = \frac{1}{2} x_2 + \frac{1}{2} x_3\} \geq \inf \{\|Tx - Ix\| : x \in C\},$$

it follows that

$$\inf \{ \|Tx - Ix\| : x \in C \} = 0.$$

Each of the sets

$$K_n = \{x \in C : \|Tx - Ix\| \leq 1/n\}, H_n = \{x \in C : \|Tx - Ix\| \leq (a+1)/an\}$$

(for $n = 1, 2, \dots$) must therefore be non-empty and obviously

$$K_1 \supseteq K_2 \supseteq \dots \supseteq K_n \supseteq \dots$$

Thus each of the sets \overline{TK}_n , where \overline{TK}_n denotes the closure of TK_n , must be non-empty for $n = 1, 2, \dots$ and

$$\overline{TK}_1 \supseteq \overline{TK}_2 \supseteq \dots \supseteq \overline{TK}_n \supseteq \dots$$

Further, for arbitrary x, y in K_n ,

$$\begin{aligned} \|Tx - Ty\| &\leq a \|Ix - Iy\| + (1-a) \max \{ \|Tx - Ix\|, \|Ty - Iy\| \} \\ &\leq a (\|Tx - Ix\| + \|Tx - Ty\| + \|Ty - Iy\|) + (1-a)/n \\ &\leq (a+1)/n + a \|Tx - Ty\| \end{aligned}$$

and so

$$\|Tx - Ty\| \leq \frac{a+1}{(1-a)n}.$$

Thus

$$\lim_{n \rightarrow \infty} \text{diam}(TK_n) = \lim_{n \rightarrow \infty} \text{diam}(\overline{TK}_n) = 0.$$

It follows, by a well known result of Cantor (see, for example [6], p. 156), that the intersection $\bigcap_{n=1}^{\infty} \overline{TK}_n$ contains exactly one point w .

Now let y be an arbitrary point in \overline{TK}_n . Then for arbitrary $\epsilon > 0$ there exists a point y' in K_r such that

$$\|Ty' - y\| < \epsilon \tag{2.4}$$

and so, using the weak commutativity of T and I and the nonexpansiveness of I , we have from (2.1) and (2.4):

$$\begin{aligned} \|Ty - Iy\| &\leq \|Ty - TIy'\| + \|TIy' - ITy'\| + \|ITy' - Iy\| \\ &\leq a \|Iy - I^2y'\| + (1-a) \max \{ \|Ty - Iy\|, \|TIy' - I^2y'\| \} \\ &\quad + \|Iy' - Ty'\| + \|Ty' - y\| \\ &\leq a \|y - Iy'\| + (1-a) \max \{ \|Ty - Iy\|, \|TIy' - ITy'\| + \|Ty' - Iy'\| \} \\ &\quad + 1/n + \epsilon \\ &\leq a (\|y - Ty'\| + 1/n) + (1-a) \max \{ \|Ty - Iy\|, 2/n \} + 1/n + \epsilon \\ &\leq (1+a) \epsilon + (a+1)/n + (1-a) \max \{ \|Ty - Iy\|, 2/n \}. \end{aligned}$$

Since ϵ is arbitrary it follows that

$$\|Ty - Iy\| \leq (a+1)/n + (1-a) \max \{ \|Ty - Iy\|, 2/n \}. \tag{2.5}$$

If $\|Ty - Iy\| \leq 2/n$, then we have

$$\|Ty - Iy\| \leq 2/n < (a+1)/an.$$

If $||Ty - Iy|| > 2/n$, (2.5) implies

$$||Ty - Iy|| \leq (a+1).n + (1-a).||Ty - Iy||.$$

So in both cases y lies in H_n . Thus $\overline{TK_n} \subseteq H_n$ and so the point w must be in H_n for $n = 1, 2, \dots$. It follows that

$$||Tw - Iw|| \leq (a+1)/an$$

for $n = 1, 2, \dots$ and so $Tw = Iw$.

Since (1.1) holds, we also have $ITw = TIw$. Thus

$$\begin{aligned} ||T^2w - Tw|| &\leq a||ITw - Iw|| + (1-a) \max \{ ||T^2w - ITw||, ||Tw - Iw|| \} \\ &= a||T^2w - Tw|| \end{aligned}$$

and it follows that $Tw = w'$ is a fixed point of T since $a < 1$. Further $Iw' = ITw + TIw = ITw = Tw' = w'$ and so w' is also a fixed point of I . Now suppose that T and I have a second common fixed point w'' . Then

$$\begin{aligned} ||w' - w''|| &= ||Tw' - Tw''|| \\ &\leq a||Iw' - Iw''|| + (1-a) \max \{ ||Tw' - Iw'||, ||Tw'' - Iw''|| \} \\ &\leq a||w' - w''|| \end{aligned}$$

and the uniqueness of the common fixed point follows since $a < 1$. This completes the proof of the theorem.

EXAMPLE 2. Let $X = \mathbb{R}$ and $C = [0, 1]$ with the usual norm. Let T and I be as in example 1. I is clearly linear and nonexpansive and further

$$TC = [0, 1/5] \subset [0, 1/2] = IC.$$

Thus

$$||Tx - Ty|| = \frac{4||x - y||}{(x+4)(y+4)} \leq \frac{1}{2} \cdot \frac{||x - y||}{2} = \frac{1}{2} ||Ix - Iy||$$

for all x, y in C and inequality (2.1) is satisfied for $a = 1/2$.

So all the assumptions of Theorem 2 hold and $w = 0$ is the unique common fixed point of T and I .

Letting I be the identity mapping in Theorem 2, we have the following corollary which extends Theorem 1:

COROLLARY. Let T be a mapping of C into itself satisfying the inequality

$$||Tx - Ty|| \leq a||x - y|| + (1-a) \max \{ ||Tx - x||, ||Ty - y|| \}$$

for all x, y in C , where $0 < a < 1$. Then T has a unique fixed point.

The result of this corollary was given in [7].

We note that the weak commutativity in Theorem 2 is a necessary condition. It suffices to consider the following example:

EXAMPLE 3. Let $X = \mathbb{R}$ and let $C = [0, 1]$ with the usual norm.

Define T and I by $Tx = 1/3$, $Ix = x/2$ for any x in C .

It is easily seen that all the conditions of Theorem 2 are satisfied except that of weak commutativity since with $x = 1/2$

$$||T(1/2) - IT(1/2)|| = 1/6 > 1/12 = ||T(1/2) - I(1/2)|| .$$

However T and I do not have a common fixed point.

We conclude that although the mappings T and I in Theorem 2 have a unique common fixed point in C , it is possible for them to have other fixed points, as proved in the next example:

Example 4. Let $X = C = \mathbb{R}^2$ with norm

$$||(x,y)|| = \max \{|x|, |y|\}$$

for all (x,y) in \mathbb{R}^2 . Define mappings T and I on \mathbb{R}^2 by

$$T(x,y) = (0,y), I(x,y) = (x,-y)$$

for all (x,y) in \mathbb{R}^2 . Then for all $(x,y) \in \mathbb{R}^2, (x',y') \in \mathbb{R}^2$

$$||T(x,y) - T(x',y')|| = |y - y'|$$

and

$$\begin{aligned} a||I(x,y) - I(x',y')|| + (1-a) \max \{||T(x,y) - I(x,y)||, ||T(x',y') - I(x',y')||\} \\ = a \max \{|x-x'|, |y-y'|\} + (1-a) \max \{|x|, 2|y|, |x'|, 2|y'|\} \\ \geq a|y - y'| + 2(1-a) \max \{|y|, |y'|\} \\ \geq a|y - y'| + (1-a) (|y| + |y'|) \\ \geq |y-y'| \end{aligned}$$

if $0 < a < 1$. Since T commutes with I and I is a linear isometry, it follows that all the conditions of Theorem 2 are satisfied but T and I each have an infinite number of fixed points.

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