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BOUNDS ON THE CURVATURE FOR FUNCTIONS WITH BOUNDED BOUNDARY ROTATION OF ORDER 1-b

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ABSTRACT. Let $V_k(1-b)$, $k \geq 2$ and $b \neq 0$ real, denotes the class of locally univalent analytic functions $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ in $D = \{z: |z| < 1\}$ such that $\int_0^{2\pi} |Re\{1 + \frac{1}{b} \frac{zf''(z)}{f'(z)}\}| d\theta < \pi k$, $z = re^{i\theta} \in D$. In this note sharp bounds on the curvature of the image of $|z| = r$, $0 < r < 1$, under a mapping f belonging to the class $V_k(1-b)$ have been obtained.

KEY WORDS AND PHRASES. Analytic Function, Univalent Functions, Functions with Bounded Boundary Rotation.
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1. INTRODUCTION.

Let A denote the class of functions $f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n$ which are analytic in $D = \{z:|z| < 1\}$. For $G \in A$, we say G belongs to the class $S(1-b)$ ($b \neq 0$ complex) if and only if $G(z)/z \neq 0$ in D and $Re\{1 + \frac{1}{b} (\frac{zG'(z)}{G(z)} - 1)\} > 0$, $z \in D$.

The class $S(1-b)$ was introduced by Nasr and Aouf in [1]. It is shown in [1] that $G \in S(1-b)$ if and only if there is a function $g \in S(0)$ such that

$$G(z) = z\,[g(z)/z]^b. \tag{1.1}$$

and for $b \neq 0$ real

$$\begin{matrix} (1+r)^{-2b} \\ (1-r)^{-2b} \end{matrix} \leq |G(z)/z| \leq \begin{matrix} (1-r)^{-2b} & (1.2) \\ (1+r)^{-2b} & (1.3) \end{matrix}$$

Let $V_k(1-b)$, $K \geq 2$ and $b \neq 0$ complex, denotes the class of functions $f \in A$ which satisfy $f(z) \neq 0$ in D and

$$\int_0^{2\pi} |Re\{1 + \frac{1}{b} \frac{zf''(z)}{f'(z)}\}| d\theta \leq \pi k, \quad z = re^{i\theta} \in D.$$

The class $V_k(1-b)$ was introduced by Nasr [2]. It was shown in [2] that $f \in V_k(1-b)$ if and only if there exist two functions $g_1, g_2 \in S(0)$ such that

$$f'(z) = \{g_1(z)/z\}^{b(k+2)/4} \Big/ \{g_2(z)/z\}^{b(k-2)/4} \tag{1.4}$$

And from (1.1) and (1.4) we deduce immediately that $f \in V_k(1-b)$ if and only if there exist two functions $G_1, G_2 \in S(1-b)$ such that

$$f'(z) = \{G_1(z)/z\}^{(k+2)/4} \Big/ \{G_2(z)/z\}^{(k-2)/4}$$

The subclasses $S(1-b)$, $V_2(1-b)$ and $V_k(1-b)$ are respectively, classes of functions starlike of order $1-b$, convex of order $1-b$ and of bounded boundary rotation of order $1-b$. We shall denote the subclasses $V_2(1-b)$ and $V_k(0)$ respectively by $C(1-b)$ and V_k.

For a locally univalent function f in D the curvature $K_r^f(z)$ at the point $w = f(z)$ for the level line, i.e. the image of the circle $|z| = r$ under the mapping f , is given by

$$K_r^f(z) = \operatorname{Re}\{1 + \frac{zf''(z)}{f'(z)}\} \Big/ |zf'(z)| \tag{1.6}$$

Let $\inf K_r^B$ and $\sup K_r^B$ denote respectively, the infimum and supremum of $K_r^f(z)$ for $|z| = r$ when f belongs to a certain subclass B of locally univalent functions in A, which is normal and compact.

The problem of estimating $K_r^f(z)$ for various classes of functions has attracted considerable attention (see [3, p.p. 599-601] for the history of this problem).

The purpose of this paper is to establish $\inf K_r^{V_k(1-b)}$ and $\sup K_r^{V_k(1-b)}$ for $b \neq o$ real.

2. STATEMENT OF RESULTS.

Set $k_1 = (k-2)/4$ and $H(r) = (1+r^2)/2r - \{\log(1+r)/(1-r)\}^{-1}$, $0 < r < 1$.

A simple calculation shows that $H(r)$ increases strictly with r and that $0 < H(r) < 1$.

THEOREM 1. If $f \in V_k(1-b)$, $b > 0$, then

$$\inf K_r^{V_k(1-b)} = \begin{cases} (1-r^2)^b/r & \text{for } k = 2,\ 0 < b < 1, \quad (2.1) \\[2mm] \dfrac{(1+r)^{2b(k_1+1)-1}}{r(1-r)^{2bk_1+1}}\{1 - bkr + (2b-1)r\} & \text{otherwise} \quad (2.2) \end{cases}$$

and

$$\sup K_r^{V_k(1-k)} = \begin{cases} \dfrac{(1-r)^{r(1-b)+b}}{r(1+r)^{r(1-b)-b}} \left\{\dfrac{e}{2r} \dfrac{(1-r)^{(1+r)^2/2r}}{(1+r)^{(1-r)^2/2r}} \log(\frac{1+r}{1-r})\right\}^{1+bk_1} & (2.2)' \\[2mm] \text{for } r - \dfrac{b(k_1+1)}{1+bk_1}(1+r) < H(r) < r + \dfrac{b(k_1+1)}{1+bk_1}(1-r) & (2.3) \\[2mm] \dfrac{(1-r)^{2b(k_1+1)-1}}{r(1+r)^{2bk_1+1}} \quad 1 + bkr + (2b-1)r^2 \quad \text{otherwise} & (2.4) \end{cases}$$

These bounds are sharp for all $0 < r < 1$.

THEOREM 2. If $f \in V_k(1-b)$, $b < 0$, then

$$\inf K_r^{V_k(1-b)} = \frac{(1-r)^{2b(k_1+1)-1}}{r(1+r)^{2bk_1+1}} \{1 + bkr + (2b-1)r^2\} \tag{2.5}$$

and

$$\sup K_r^{V_k(1-b)} = \begin{cases} \dfrac{(1-r)^{r(1-b)+b}}{r(1+r)^{r(1-b)-b}} \left\{\dfrac{e}{2r} \dfrac{(1-r)^{(1+r)^2/2r}}{(1+r)^{(1-r)^2/2r}} \log\left(\dfrac{1+r}{1-r}\right)\right\}^{b(k_1+1)-1} & (2.6) \\ \text{for } r - \dfrac{bk_1}{b(k_1+1)-1}(1+r) \le H(r) \le r + \dfrac{bk_1}{b(k_1+1)-1}(1-r) & (2.7) \\ \dfrac{(1+r)^{2b(k_1+1)-1}}{r(1-r)^{2bk_1+1}} \{1 - bkr + (2b-1)r^2\} \quad \text{otherwise.} & (2.8) \end{cases}$$

These bounds are sharp for all $0 < r < 1$.

Indeed, if $k = 2$, $b > 0$ our results coincide with the results given for $\inf K_r^{c(1-b)}$ and $\sup K_r^{c(1-b)}$ by Singh [7], also for $k = 2$, $0 < b \le 1$, our results are reduced to those given for $\inf K_r^{c(1-b)}$ and $\sup K_r^{c(1-b)}$ by Zederkiewicz [4] and those given by Eenigenburg [5] for $\inf K_r^{c(1-b)}$. Moreover, for $k = 2$, $b = 1$, coincide with those given for $\inf K_r^{c(0)}$ and $\sup K_r^{c(0)}$ by Zmorovič [6] and those given by Keogh [7] for $\inf K_r^{c(0)}$. Finally, if $b = 1$ our results agree with those reached by Noonan [8] and Singh [9] for $\inf K_r^{V_k}$ and $\sup K_r^{V_k}$. But to the best of our knowledge the values of $\inf K_r^{c(1-b)}$ and $\sup K_r^{c(1-b)}$ for $b < 0$ and also the values of $\inf K_r^{V_k(1-b)}$ and $\sup K_r^{V_k(1-b)}$ for $b < 0$ and also the values of $\inf K_r^{V_k(1-b)}$ and $\sup K_r^{V_k(1-b)}$ for $b \ne 1$ are not known as yet.

3. PROOFS.

We need the following:

LEMMA 1[9]: If $g \in S(0)$, $z = r e^{i\theta} \in D$, then

$$(1-r^2)\left|\frac{g(z)}{z}\right| \le \operatorname{Re} \frac{zg'(z)}{g(z)} \le \frac{1+r}{1-r} + \frac{2r \log|(1-r^2)g(z)/z|}{(1-r^2)\log\{(1+r)/(1-r)\}} \tag{3.1}$$

Both sides of the above inequality are sharp.

COROLLARY 1. If $G \in S(1-b)$, $z = re^{i\theta} \in D$, then

$B(r, G(z)/z)$		$A(r,G(z)/z)$	for $b > 0$	(3.2)
	$\le \operatorname{Re} \dfrac{zG'(z)}{G(z)} \le$			
$A(r,G(z)/z)$		$B(r,G(z)/z)$	for $b < 0$	(3.3)

where

$$A(r,x) = \frac{1 + (2b - 1)r}{1 - r} + \frac{2r \log |(1-r)^{2b} x|}{(1-r^2)\log[(1+r)/(1-r)]}$$

and

$$B(r,x) = (1-b) + b(1-r^2) |x|^{1/b}$$

PROOF. The proof will follows immediately from (1.1) and (3.1)

PROOF OF THEOREM 1. Set

$$|G_1(z)/z| = u \text{ and } |G_2(z)/z| = v . \tag{3.4}$$

Then from (2.2), we find that u and v lie in the interval

$$[1/(1+r)^{2b}, 1/(1-r)^{2b}] . \tag{3.5}$$

In view of (1.5) and (1.6) we need to find the extreme values of

$$K_r^f(z) = V^{k_1}[(k_1 + 1) \frac{zG_1'(z)}{G_1(z)} - k_1 \frac{zG_2'(z)}{G_2(z)}] / ru^{k_1+1} , \tag{3.6}$$

In view of (3.2) and (3.6) we need to obtain the minimum of

$$F(u,v) = V^{k_1} [(k_1 + 1)B(r,u) - k_1 A(r,v)] / ru^{k_1 +1} \tag{3.7}$$

and the maximum of

$$H(u,v) = V^{k_1} [(k_1 + 1)A(r,u) - k_1B(r,v)] / ru^{k_1 +1} \tag{3.8}$$

when u and v lie in the interval given by (3.5).

This reduces the problem to finding extreme values of functions of two variables.

It is easily seen that for $0 < b \le 1$ and $k = 2$ $(k_1 = 0)$ the minimum is attained for

$$u = 1/(1-r^2)^b \tag{3.9}$$

and because the value of u lies within the interval given by (3.5) this gives the minimum. We thus obtain (2.1). If $k = 2, b > 1$ or $k > 2, b > 0$ it is readily confirmed that the roots of

$$\frac{\partial F}{\partial u} = 0 = \frac{\partial F}{\partial v}$$

do not give the minimum. Hence, the minimum is attained on the boundary for $u = 1/(1-r)^{2b}$ and $v = 1/(1-r)^{2b}$. This yields (2.2) .

In order to maximize $H(u,v)$, it is found that the equations

$$\frac{\partial H}{\partial u} = 0 = \frac{\partial H}{\partial v}$$

give

$$v = \{2r/(1-r^2)^2 \log [(1+r)/(1-r)]\}^b \tag{3.10}$$

and

$$\log [(1-r)^{2b} U] = \frac{1 + bk_1}{1 + k_1} - [\frac{1 + bk_1}{1 + k_1} + \frac{2br}{1-r}] \frac{(1-r^2)}{2r} \log (\frac{1-r}{1-r}) \tag{3.11}$$

The value of v given by (3.10) lies in the interval given by (3.5) because

$$2r/(1+r)^2 \le \log[(1+r)/(1-r)] \le 2r/(1-r)^2 \tag{3.12}$$

but the value of u given by (3.11) lies in the interval given by (3.5) if

$$r - [b(1+k_1)(1+r)/(1+bk_1)] \le H(r) \le r + [b(1+k_1)(1-r)/(1+bk_1)]. \tag{3.13}$$

This gives (2.3). When (3.13) does not hold, the maximum values of (3.8) is obtained on the boundary for $u = 1/(1-r)^{2b}$ and $v = 1/(1+r)^{2b}$. This yields (2.4) .

The above inequalities are sharp and the extremal functions are given below where equality in each case, is attained at $z = r$.

(i) For equality in (2.1)

$$f_1'(z) = 1/[(1 - ze^{it})^{\lambda}\ (1 - ze^{-it})^{(1-\lambda)}]^b\ ,\ 0 \le \lambda \le 1\ ,\ 0 < b \le 1\ ,\ \text{where}$$

$$\cos t = r \quad \text{and} \quad \lambda b = \frac{1+r^2}{r} + (1-b)r - H(r) \tag{3.14}$$

(ii) For equality in (2.2)

$$f_2'(z) = (1-z)^{2bk_1} \big/ (1+z)^{2b(1+k_1)} \tag{3.15}$$

(iii) For equality in (2.3)

$$f_3'(z) = (1-ze^{-it})^{2bk_1}\ \frac{(1-z)^{(1+bk_1)H(r)+r(b-1)-b(1+k_1)}}{(1+z)^{(1+bk_1)H(r)+(b-1)+b(1+k_1)}} \tag{3.16}$$

where

$$1+r^2 -2r\cos t = (1-r^2)^2 \big/ [1+r^2-2rH(r)], \tag{3.17}$$

(iv) For equality in (2.4)

$$f_4'(z) = (1+z)^{2bk_1} \big/ (1-z)^{2b(k_1+2)} \tag{3.18}$$

PROOF OF THEOREM 2. Taking into consideration (1.3), (1.5), (1.6), (3.3) and using the notation $G_1(z)/z = u$ and $G_2(z)/z = v$, we find that for u and v lie in the interval

$$[1/(1-r)^{2b},\ 1/(1+r)^{2b}], \tag{3.19}$$

we need to obtain the minimum of

$$F_1(u,v) = v^{k_1}[(1+k_1)A(r,u)-k_1B(r,v)]/ru^{1+k_1} \tag{3.20}$$

and the maximum of

$$H_1(u,v) = V^{k_1}(1+k_1)B(r,u)-k_1\ A(r,v)]/ru^{1+k_1} \tag{3.21}$$

It is readily verified in the case of (3.20) that the equations:

$$\frac{\partial F_1}{\partial u} = 0 = \frac{\partial F_1}{\partial v}$$

do not give the minimum and that minimum is attained for $u = 1/(1-r)^{2b}$ and $v = 1/(1+r)^{2b}$ and this value is given by (2.5). Simple calculation confirms the case of equality for the function $f(z) = f_4(z)$ given by (3.18).

In order to maximize $H_1(u,v)$ given by (3.21), it is found that the equations:

$$\frac{\partial H_1}{\partial u} = 0 = \frac{\partial H_1}{\partial v}$$

give

$$v = [2r/(1-r^2)^2 \log (1+r)/(1-r)]^b \tag{3.22}$$

and

$$\log (1-r)^{2b} u = \frac{1+bk_1}{1+k_1} - [\frac{1+bk_1}{1+k_1} + \frac{2br}{1-r}] \frac{(1-r^2)}{2r} \log (\frac{1+r}{1-r}) \tag{3.23}$$

The value of v given by (3.22) lies in the interval (3.19) because,

$1 > H(r) > 0$

but the value of u given by (3.23) lies in the interval (3.19) if

$$r-[b(k-2)(1+r)/(b(k+2)-4)] < H(r)+[b(k-2)(1-r)/(b(k+2)-4)] \tag{3.25}$$

This proves (2.6). The case of equality can be directly confirmed by the function $f(z)$ given by

$$f_5'(z) = \frac{(1+z)^{\frac{H(r)b(1+k_1)+bk_1+r(1-b)}{H(r)b(1+k_1)-bk_1+r(1-b)}}}{(1-z)} (1-ze^{-it})^{-2b(1+k_1)} \tag{3.26}$$

where

$$1+r^2-2r \cos t = (1-r^2)^2/[1+r^2-2rH(r)] \tag{3.27}$$

when (3.15) does not hold, the maximum value of $H_1(u,v)$ is attained for $u = 1/(1+r)^{2b}$, $v = 1/(1-r)^{2b}$ and the corresponding value of $\sup K_r^{V_k(1-b)}$ is given by (2.7). Simple calculations confirm the case of equality for the functions given by (3.15).

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