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BANACH SPACES OF FUNCTIONS ANALYTIC IN A POLYDISC

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ABSTRACT. This paper is concerned with functions of several complex variables analytic in the unit polydisc. Certain Banach spaces to which these functions might belong are defined and some relationships between them are developed. The space of linear functionals for the Banach space of functions analytic in the open unit polydisc and continuous on the unit torus is then described in terms of analytic functions using an extension of the Hadamard product.

KEY WORDS AND PHRASES. Banach space, linear functional, analytic function.
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1. INTRODUCTION.

In 1950 A. E. Taylor [1] studied Banach spaces of functions analytic in the unit disc. One of his principal results was a representation of linear functionals in terms of functions analytic in the unit disc. In this paper, the results of Taylor are extended to functions analytic in the unit polydisc in n-dimensional complex space. The goal is the representation theorem for linear functions, Theorem 4.5, in which the functionals are expressed in terms of a Hadamard product. Taylor's results have proved to be very useful in work involving certain singular linear differential operators. His representation theorem for linear functionals was a key part of the work of Grimm and Hall in [2], and of subsequent work of Hall in [3] and [4].

The following notation will be used extensively:

U = open unit disc in the complex plane C, (1.1)

T = unit circle in C, (1.2)

$T_\varepsilon = \{z \in C: |z| = 1-\varepsilon\}$, $0 < \varepsilon < 1$, (1.3)

Z_+ = nonnegative integers, (1.4)

$U^n = U \times U \times\ldots\times U$, (n copies of U) the unit polydisc in C^n, (1.5)

$T^n = T \times T \times\ldots\times T$, (n copies of T) the unit torus in C^n, (1.6)

$T^n_\varepsilon = T_\varepsilon \times T_\varepsilon \times\ldots\times T_\varepsilon$, (n copies of T_ε), (1.7)

$Z^n_+ = Z_+ \times Z_+ \times\ldots\times Z_+$ (n copies of Z_+). (1.8)

If $z = (z_1,z_2,\ldots,z_n)$ and $w = (w_1,w_2,\ldots,w_n)$ are in C^n, define zw by $zw = (z_1w_1,z_2w_2,\ldots,z_nw_n)$. Further, if $z \in C^n$ and $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in Z_+^n$ define z^α by $z^\alpha = z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n}$ and $|\alpha|$ by $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. Denote by $\mathcal{A}^n$ the class of functions on C^n which are analytic in U^n and define the function $u_\alpha \in \mathcal{A}^n$, for each $\alpha \in Z_+^n$, by $u_\alpha(z) = z^\alpha$. Also define the operators $\mathcal{U}_x : \mathcal{A}^n \to \mathcal{A}^n$ and $\mathcal{T}_w\, \mathcal{A}^n \to \mathcal{A}^n$ by

$$\mathcal{U}_x f = g, \text{ where } x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n, \text{ and} \tag{1.9}$$

$$g(z) = f(ze^{ix}) \equiv f(z_1e^{ix_1},z_2e^{ix_2},\ldots,z_ne^{ix_n}), \tag{1.10}$$

$$\mathcal{T}_w f = g, \text{ where } w \in \overline{U^n}, \text{ and } g(z) = f(wz). \tag{1.11}$$

The operators $\mathcal{U}_x$ and $\mathcal{T}_w$ are easily seen to be linear.

If $f \in \mathcal{A}^n$ then f has the power series expansion

$$f(z) = \sum_{\alpha \in Z_+^n} f_\alpha z^\alpha. \tag{1.12}$$

Denote f_α by $\gamma_\alpha(f)$ where γ_α is the functional defined by

$$\gamma_\alpha(f) = (\alpha_1!\alpha_2!\cdots\alpha_n!)^{-1} \left.\frac{\partial^{|\alpha|}f}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}\right|_{z=(0,\ldots,0)}. \tag{1.13}$$

If f and g are in $\mathcal{A}^n$ with power series

$$f(z) = \sum_{\alpha \in Z_+^n} f_\alpha z^\alpha \text{ and } g(z) = \sum_{\alpha \in Z_+^n} g_\alpha z^\alpha, \tag{1.14}$$

define the Hadamard product of f and g by

$$H(f,g;z) = \sum_{\alpha \in Z_+^n} f_\alpha g_\alpha z^\alpha. \tag{1.15}$$

$H(f,g;z)$ is clearly in $\mathcal{A}^n$, and the following also can be proved:

$$\gamma_\alpha(\mathcal{T}_w f) = w^\alpha \gamma_\alpha(f), \tag{1.16}$$

$$H(f,g;z) = H(g,f;z), \tag{1.17}$$

$$H(af+bg,h;z) = aH(f,h;z) + bH(g,h;z), \tag{1.18}$$

$$H(\mathcal{T}_w f,g;z) = H(f,g;wz), \tag{1.19}$$

$$H(f,g;z) = \left(\frac{1}{2\pi i}\right)^n \int_{T_{\varepsilon_1}\times\cdots\times T_{\varepsilon_n}} f(\xi_1,\ldots,\xi_n)\, g\left(\frac{z_1}{\xi_1},\ldots,\frac{z_n}{\xi_n}\right)\frac{d\xi_1}{\xi_1}\cdots\frac{d\xi_n}{\xi_n}, \tag{1.20}$$

where $|z_i| < |\xi_i|$ for $i=1,\ldots,n$.

Equation (1.20) can also take the form

$$H(f,g;z) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\!\!\cdots\int_0^{2\pi} f(r_1e^{i\theta_1},\ldots,r_ne^{i\theta_n})\, g\left(\frac{z_1}{r_1}e^{-i\theta_1},\ldots,\frac{z_n}{r_n}e^{-i\theta_n}\right)d\theta_1\cdots d\theta_n \tag{1.21}$$

where $r_i = 1 - \varepsilon_i$ and $z \in U^n$.

2 SPACES OF TYPE A_k.

Let B be a nontrivial complex Banach space, each element of which belongs to A^n. Such a space will be called a space of type A. B may also have one or more of the following properties.

P_1: There exists a constant A such that $|\gamma_\alpha(f)| \leq A||f||$ if $f \in B$ and $\alpha \in Z^n_+$. The least such constant will be denoted $A_1(B)$.

P_2: $u_\alpha \in B$ for all $\alpha \in Z^n_+$. There exists a constant A such that $||u_\alpha|| \leq A$ for all α. The least such constant will be denoted $A_2(B)$.

P_3: $U_x f \in B$ if $f \in B$ and x is a real n-tuple. Also $||U_x f|| = ||f||$.

P_4: $T_r f \in B$ if $f \in B$ and $r = (r_1, r_2, \ldots, r_n)$ with $0 \leq r_i < 1$. There exists a constant A such that $||T_r f|| \leq A||f||$. The least such constant will be denoted $A_4(B)$.

B will be called a space of type A_k if B is of type A and also satisfies axioms $P_1, \ldots, P_k$.

Let B* denote the space of continuous linear functionals on B. Then by P_1, $\gamma_n \in B^*$ and it can be shown that

$$A_1(B) = \sup_\alpha ||\gamma_\alpha||, \text{ and} \tag{2.1}$$

$$A_2(B) = \sup_\alpha ||u_\alpha||. \tag{2.2}$$

Other relations satisfied by the constants $A_k(B)$ are

$$A_4(B) = \sup_r ||T_r||, \tag{2.3}$$

$$1 \leq A_1(B) A_2(B), \tag{2.4}$$

$$1 \leq A_4(B) \tag{2.5}$$

Let B be a space of type A_1. For any fixed $g \in A^n$ and $z \in U^n$ define

$$N(g;z) = \sup_{||f||=1} |H(f,g;z)|. \tag{2.6}$$

The following are clear from the properties of the Hadamard product.

$$N(g+h;z) \leq N(g;z) + N(h;z) \tag{2.7}$$

$$N(ag;z) = |a| N(g;z) \tag{2.8}$$

$$N(T_w g;z) = N(g;wz), \ |w| < 1. \tag{2.9}$$

THEOREM 2.1. Let B be a space of type A_3. Then the function N(g;z) has the properties:

$N(g,z) = N(g,\hat{z})$, where $g \in A^n$, $z \in U^n$ and $\hat{z} = (|z_1|, |z_2|, \ldots, |z_n|)$, (2.10)

$N(g;(r_1, \ldots, r_n))$ is a nondecreasing function of each r_i, $i=1,\ldots,m$, (2.11)

where $0 \leq r_i < 1$,

$N(g;(r_1,\ldots,r_n)) = 0$ for all $(r_1,\ldots,r_n)$ if and only if $g \equiv 0$. (2.12)

PROOF. Let $\mathcal{M}$ denote the set of all nonzero elements of B and define, for $f \in \mathcal{M}$

$$M(f;z) = \frac{H(f,g;z)}{\|f\|} \tag{2.13}$$

where g is fixed in $\mathcal{A}^m$. Also define

$$M(z) = \sup_{f \in \mathcal{M}} |M(f;z)|. \tag{2.14}$$

Observe that $M(z) = N(g;z)$ and also, for $x \in R^n$ and $z \in U^n$

$$M(\mathcal{U}_x f;z) = M(f;ze^{ix}). \tag{2.15}$$

Let $z \in U^n$ be fixed. Then

$$\begin{aligned} |M(f;z)| &= \left|H\left(\frac{f}{\|f\|},g;z\right)\right| \\ &= \frac{1}{\|f\|}|H(f,g;z)| \\ &\leq \frac{1}{\|f\|} \sum_\alpha |f_\alpha||g_\alpha|\ |z^\alpha| \\ &\leq \frac{1}{\|f\|} \sum_\alpha A_1(B)\|f\|\ |g_\alpha|\ |z^\alpha| \\ &= A_1(B) \sum_\alpha |g_\alpha|\ |z^\alpha|. \end{aligned} \tag{2.16}$$

Hence, for fixed $z \in U^n$, $|M(f;z)|$ is bounded as f varies over $\mathcal{M}$.

Now for $x \in R^n$, $\mathcal{U}_x f$ varies over all of $\mathcal{M}$ as f varies over all of $\mathcal{M}$ since $f = \mathcal{U}_x(\mathcal{U}_x f)$, and so by (2.15) we get $M(z) = M(\hat{z})$ if x is chosen carefully, and (2.10) is proved.

Next, let $R = (R_1,\ldots,R_i,\ldots,R_n)$ and $r = (r_1,\ldots,r_i,\ldots,r_n)$ with $0 \leq R_k = r_k < 1$ for $k=1,\ldots,n$, $k \neq i$ and $0 \leq r_i < R_i < 1$. Also suppose $f \in \mathcal{M}$. Then let z_R and z_r be the points on the tori $\{z\colon |z_k| = R_k\}$ and $\{z\colon |z_k| = r_k\}$, respectively, at which $|M(f;z)|$ assumes its maximum value. The maximum principle for polydisc functions now yields

$$|M(f;z_r)| \leq |M(f;z_R)|. \tag{2.17}$$

But $|M(f;z_R)| \leq M(z_R) = M(\hat{z}_R) = M(R)$ and so

$$|M(f;r)| \leq |M(f;z_r)| \leq M(R), \tag{2.18}$$

which implies $M(r) \leq M(R)$, and (2.11) is proved.

Finally, if $g \equiv 0$, $N(g,r) \equiv 0$ for all r. If $g \not\equiv 0$, then $\gamma_\alpha(g) \neq 0$ for some α. Let $f = u_\alpha$ for this α. Then $H(f,g;r) = \gamma_\alpha(g) r_1^{\alpha_1} \cdots r_n^{\alpha_n}$ which is nonzero if $r_k \neq 0$, $k=1,\ldots,n$, and the proof is complete.

3. THE SPACES B' AND B°.

Suppose B is a space of type A_3. Two related spaces which will be used later in characterizing B* will be defined as follows:

$$B' = \{F \in \mathcal{A}^n\colon N(F;r) \text{ is uniformly bounded in } r,\ 0 \leq r_k < 1,\ k=1,\ldots,n\}, \tag{3.1}$$

$$B^0 = \{F\varepsilon A^n : \lim_{r\to(1,\ldots,1)} H(f,F;r) \text{ exists for each } f \varepsilon B\}. \tag{3.2}$$

For $F \varepsilon B^1$ define $N(F) = \sup\{N(f;r): 0 \le r_k < 1, k=1,\ldots,n\}$. By (2.11) we have

$$N(F) = \lim_{r\ (1,\ldots,1)} N(F;r) \tag{3.3}$$

THEOREM 3.1. If B is a space of type A_3 then B' is a Banach space with norm N(F).

PROOF. From the properties of the Hadamard product and N(F;r), and from theorem 2.1, B' is clearly a normed linear space with norm N(F). To prove completeness proceed as follows.

Let $\{F_j\}$ be a Cauchy sequence in B'. Then there is a $J > 0$ such that if $j > J$, $N(F_j) - N(F_J) < 1$, and $N(F_j)$ is bounded by, say, E. Let $0 \le r_k < 1$ for $k=1,\ldots,n$ and consider, for $\alpha \varepsilon Z^m_+$,

$$\begin{aligned}|\gamma_\alpha(F_j-F_m)r^\alpha| &= H(u_\alpha,F_j-F_m;r)| \\ &\le ||u_\alpha||N(F_j-F_m) \le A_2(B)N(F_j-F_m).\end{aligned} \tag{3.4}$$

Thus, $\{\gamma_\alpha(F_j)\}$ is also a Cauchy sequence for each $\alpha \varepsilon Z^n_+$. Now define $a_\alpha = \lim_{j\to\infty} \gamma_\alpha(F_j)$ and note that convergence is uniform in α and that the sequence $\{a_\alpha\}$, $\alpha \varepsilon Z^n_+$, is bounded. Using the above argument with $F_m = 0$ we obtain $|\gamma_\alpha(F_j)r^\alpha| \le A_2(B)E$ which yields $|a_\alpha| \le A_2(B)E$. Next define

$$F(z) = \sum_{\alpha\varepsilon Z^n_+} a_\alpha z^\alpha. \tag{3.5}$$

Clearly $F \varepsilon A$, and it will be shown that $F \varepsilon B'$ and $\lim_{j\to\infty} F_j = F$.

Consider, for fixed r, and $f \varepsilon A$,

$$|H(f,F_j;r) - H(f,F;r)| \le \sum_\alpha |\gamma_\alpha(f)r^\alpha \sup_\alpha|\gamma_\alpha(F_j) - a_\alpha|. \tag{3.6}$$

Since $\lim_{j\to\infty}[\gamma_\alpha(F_j)-a_\alpha] = 0$ uniformly in α,

$$\lim_{j\to\infty} H(f,F_j;r) = H(f,F;r). \tag{3.7}$$

Further, for $f \varepsilon B$

$$|H(f,G_j;r)| \le ||f||N(F_j) \le E||f|| \tag{3.8}$$

which implies $|H(f,F;r)| \le E||f||$ or $N(F;r) \le E$. Hence, $F \varepsilon B'$.

Finally, for $\varepsilon > 0$, choose $j_0(\varepsilon)$ such that if $j,m \ge j_0(\varepsilon)$ then $N(F_j-F_m) < \varepsilon$. For $f \varepsilon B$ (3.5) yields

$$|H(f,F_j-F_m;r)| \le N(F_j-F_m)||f|| < \varepsilon||f||. \tag{3.9}$$

As $m \to \infty$ (3.4) yields $|H(f,F_j-F;r)| \le \varepsilon||f||$, if $j \ge j_0(\varepsilon)$, which means $N(F_j-F) \le \varepsilon$ and the proof is complete.

THEOREM 3.2. If B is a Banach space of type A_3 then B^0 is a linear subset of B' and is a Banach space of type A_4 with norm N(F).

PROOF: Clearly B^0 is a linear subset of $\mathcal{A}$. For $f \varepsilon B$ and $F \varepsilon B^0$ let $\psi_r(f) = H(f,F;r)$ define a functional of f over B. Then

$$\sup_{||f||=1} |\psi_r(f)| = N(F;r), \tag{3.10}$$

and since $F \varepsilon B^0$ then $N(f;r)$ is bounded as a function of r by the uniform boundedness principle. Thus, B^0 is a linear subset of B'.

In order to show that B^0 is closed in B', let $\{F_j\} \subset B^0$ and $F \varepsilon B'$ be such that $\lim_{j\to\infty} N(F_j-F) = 0$ For $f \varepsilon B$

$$\begin{aligned} |H(f,F;r)-H(f,F_j;\rho)| &\le |H(f,F-F_j;r)| + |H(f,F_j;r)-H(f,F_j;\rho)| + |H(f,F_j-F;\rho)| \\ &\le 2N(F_j-F)||f|| + |H(f,F_j;r)-H(f,F_j;\rho)|. \end{aligned} \tag{3.11}$$

For $f \varepsilon B$ fixed and $\varepsilon > 0$ choose j large enough so that $2N(F_j-F)||f|| < \varepsilon/2$ and choose r and ρ close enough to $(1,1,\ldots,1)$ so that $|H(f,F_j;r)-H(f,F_j;\rho) < \varepsilon/2$. Therefore $\lim_{r,\rho\to(1,\ldots,1)} |H(f,F;r)-H(f,F;\rho)| = 0$ and $F \varepsilon B^0$.

4. REPRESENTATION OF LINEAR FUNCTIONALS.

In this section a series of results which culminate in a representation of the space B^* and its relationship to the spaces B' and B^0 will be proved.

THEOREM 4.1. Let B be a Banach space of type $\mathcal{A}_4$. If $\gamma \varepsilon B^*$ let

$$G(z) = \sum_{\alpha\varepsilon Z^n_+} \gamma(u_\alpha) z^\alpha, \quad z \varepsilon U^n. \tag{4.1}$$

Then $G \varepsilon B'$ and $||G|| = N(G) \le A_4(B)||\gamma||$.

PROOF: $G \varepsilon \mathcal{A}^n$ since $|\gamma(u_\alpha)| \le ||\gamma|| A_2(B)$. Also, $\gamma(u) = \gamma_\alpha(G) = G_\alpha$. Let $f \varepsilon \mathcal{A}^n$ and $w \varepsilon U^n$. Then $\mathcal{T}_w f \varepsilon B$ and

$$\gamma(\mathcal{T}_w f) = \sum_\alpha f_\alpha \gamma(z^\alpha) w^\alpha = \sum_\alpha f_\alpha G_\alpha w^\alpha = H(f,G;w). \tag{4.2}$$

If $f \varepsilon B$ then

$$|H(f,G;r)| = |\gamma(\mathcal{T}_r f)| \le A_4(B)||\gamma||\ ||f||. \tag{4.3}$$

Hence $N(G;r) = \sup_{||f||=1} |H(f,G;r)| \le A_4(B)||\gamma||$ and $G \varepsilon B'$. Also,

$||G|| = N(G) = \sup_r N(G;r) \le A_4(B)\ ||\gamma||$.

The passage from γ to G given by (4.1) defines an operator $\Gamma\colon B^* \to B'$ where $\Gamma(\gamma) = G$.

THEOREM 4.2. Let B be a Banach space of type $\mathcal{A}_4$. Then Γ is a linear operator and $||\Gamma|| = A_4(B)$. Γ has an inverse if and only if the linear subspace of B spanned by $\{u_\alpha\}$, $\alpha \varepsilon Z^n_+$, is dense in B.

PROOF. By Theorem 4.1 Γ is linear and $||\Gamma|| \le A_4(B)$. By (4.2)

$$|\gamma(\mathcal{T}_r f)| \le ||f||\ ||G|| \le ||f||\ ||\Gamma||\ ||\gamma||. \tag{4.4}$$

Now choose γ so that $||\gamma|| = 1$ and $|\gamma(\mathcal{T}_r f)| = ||\mathcal{T}_r f||$ so that $||\mathcal{T}_r f|| \le ||f||\ ||\Gamma||$.

Then $||T_r|| \le ||\Gamma||$ and $A_4(B) \le ||\Gamma||$. Hence $||\Gamma|| = A_4(B)$.

Suppose $\Gamma(\gamma) = 0$. Then $\gamma(u_\alpha) = 0$ for all $\alpha \in Z^n_+$ and hence saying that Γ^{-1} exists if and only if $\gamma(u_\alpha) = 0$ for all $\alpha \in Z^n_+$ is equivalent to saying $\gamma = 0$. Thus Γ^{-1} exists if and only if $\{u_\alpha\}$ is total in B which is equivalent to the subspace spanned by $\{u_\alpha\}$ being dense in B.

For $f \in B$, $H(f,F;r)$ defines a linear functional with norm $N(F;r)$, so if $F \in B^0$

$$\gamma(f) = \lim_{r\to(1,\dots,1)} H(f,F;r) \tag{4.5}$$

defines an element of B^* corresponding to F. The passage from F to γ given by (4.5) defines an operator $\Lambda : B^0 \to B^*$ where $\Lambda(F) = \gamma$.

THEOREM 4.3. Let B be a Banach space of type A_4. Then Λ is a linear operator,

$$\Gamma\Lambda(F) = F, \text{ for } F \in B^0, \text{ and} \tag{4.6}$$

$$||\Lambda(F)|| \le N(F) \le A_4(B)||\Lambda(F)||, \text{ for } F \in B^0. \tag{4.7}$$

Thus Λ defines an injective mapping, with bounded inverse, of B^0 onto a subspace of B^*.

PROOF: Let $F \in B^0$, $\Lambda(F) = \gamma$ and $\Gamma(\gamma) = G$. Then

$$\gamma_\alpha(G) = \gamma(u_\alpha) = \lim_{r\to(1,\dots,1)} H(u_\alpha,F;r) = \lim_{r\to(1,\dots,1)} \gamma_\alpha(F)r^\alpha = \gamma_\alpha(F), \text{ and so } \Gamma(\Lambda(F)) = F.$$

Also, since $|H(f,F;r)| \le ||f||\ ||F||$ it follows that $|\gamma(f)| \le ||f||\ ||F||$ where $\gamma = \Lambda(F)$. Thus, from (4.6)

$$||F|| = ||\Gamma(\Lambda(F))|| \le ||\Gamma||\ ||\Lambda(F)|| = A_4(B)||\Lambda(F)||. \tag{4.8}$$

If the space B satisfies an additional axiom, viz.,

P_5 : If $f \in B$ then $T_r f \in B$ and $\lim_{r\to(1,\dots,1)} ||T_r f-f|| = 0$,

then B^*, B' and B^0 turn out to be isometrically isomorphic.

THEOREM 4.4. Let B be a Banach space of type A_4 also satisfying P_5. Then

$$B^0 = B' \tag{4.9}$$

$$\Lambda \text{ is bijective and isometric, and} \tag{4.10}$$

$$\Lambda^{-1} = \Gamma. \tag{4.11}$$

PROOF: By (1.19) $H(T_r f,F;\rho) = H(T_\rho f,F;r)$ and so

$$H(f,F;r) - H(f,F;\rho) = H(f-T_\rho f,F;r) + H(T_r f-T_\rho f,F;\rho) + H(T_\rho f-f,F;\rho). \tag{4.12}$$

Therefore, if $f \in B$ and $F \in B'$

$$|H(f,F;r) - H(f,F;\rho)| \le [2||f-T_\rho f|| + ||T_r f-T_\rho f||]N(F), \tag{4.13}$$

and by P_5

$$\lim_{r,\rho\to(1,\dots,1)} |H(f,F;r) - H(f,F;\rho)| = 0, \tag{4.14}$$

so $F \in B^0$. Since $B^0 \subset B'$ already, this means $B' = B^0$.

From P_5, $\lim_{r\to(1,\ldots,1)} \|T_r f\| = \|f\|$. If $|w_i| = r_i < 1$, $i=1,\ldots,n$, and $T_w f$ is analytic in w then $\|T_w f\| = \|T_r f\|$ by P_3. Hence from the maximum modulus theorem for functions with values in a Banach space (see [3], p. 211) $\|T_r f\|$ is a nondecreasing continuous function of r. This yields $A_4(B) = 1$ and by Theorem 4.3 Λ is isometric.

Now suppose $f \varepsilon B$ and $\gamma \varepsilon B^*$. Then

$$\begin{aligned}[\Lambda(\Gamma(\gamma))](f) &= \lim_{r\to(1,\ldots,1)} H(f,\Gamma(\gamma);r) \\ &= \lim_{r\to(1,\ldots,1)} \gamma(T_r f) = \gamma(f).\end{aligned} \tag{4.15}$$

As a corollary of Theorem 4.4 the following theorem provides a representation of B* in terms of the Hadamard product.

THEOREM 4.5. Under the hypotheses of Theorem 4.4 every linear functional $\gamma \varepsilon B^*$ is representable in the form

$$\begin{aligned}\gamma(f) &= \lim_{r\to(1,\ldots,1)} H(f,F;r) = \\ &\lim_{r\to(1,\ldots,1)} \left(\frac{1}{2\pi i}\right)^n \int_{T_{\varepsilon_1}\times\ldots\times T_{\varepsilon_n}} f(\xi_1,\ldots,\xi_n)F\left(\frac{r_1}{\xi_1},\ldots,\frac{r_n}{\xi_n}\right)\frac{d\xi_1}{\xi_1}\cdots\frac{d\xi_n}{\xi_n},\end{aligned} \tag{4.16}$$

where, $0 < \varepsilon_i < 1$, $0 < r_i < \xi_i < 1$, $i=1,\ldots,n$ and $F \varepsilon B^0$. F uniquely determines and is uniquely determined by γ and $\|\gamma\| = N(F)$.

Equation (4.16) can be expressed in terms of real integrals using (1.21).

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Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

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