

## ON CERTAIN CLASSES OF $p$ -VALENT FUNCTIONS

**M. K. AOUF**

Department of Mathematics  
 Faculty of Science  
 University of Mansoura  
 Mansoura, Egypt

(Received February 20, 1985)

ABSTRACT. Let  $V_k^\lambda(\alpha, b, p)$  ( $k \geq 2$ ,  $b \neq 0$  is any complex number,  $0 \leq \alpha < p$  and  $|\lambda| < \pi/2$ ) denote the class of functions  $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$  analytic in  $U = \{z: |z| < 1\}$  having  $(p-1)$  critical points in  $U$  and satisfying

$$\limsup_{r \rightarrow 1^-} \int_0^{2\pi} \left| \frac{\operatorname{Re}\{e^{i\lambda} [p + \frac{1}{b} (1 + \frac{zf''(z)}{f'(z)} - p)] - \alpha \cos \lambda\}}{p - \alpha} \right| d\theta \leq k\pi \cos \lambda.$$

In this paper we generalize both those functions  $f(z)$  which are  $p$ -valent convex of order  $\alpha$ ,  $0 \leq \alpha < p$ , with bounded boundary rotation and those  $p$ -valent functions  $f(z)$  for which  $zf'(z)/p$  is  $\lambda$ -spirallike of order  $\alpha$ ,  $0 \leq \alpha < p$ .

KEY WORDS AND PHRASES.  $p$ -Valent, starlike, convex, spirallike functions, functions with bounded boundary rotation.

AMS SUBJECT CLASSIFICATION CODE. 30A32, 30A36.

### 1. INTRODUCTION.

Let  $M_k$  ( $k \geq 2$ ) denote the lower class of real-valued functions  $m(t)$  of bounded variation on  $[0, 2\pi]$  which satisfy the conditions  $\int_0^{2\pi} dm(T) = 2$  and  $\int_0^{2\pi} |dm(t)| \leq k$ .

Let  $A_p$ , where  $p$  is a positive integer, denote the class of functions  $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$  which are analytic in  $U = \{z: |z| < 1\}$ . For  $f \in A_p$ , we say that  $f$  belongs to the class  $V_k^\lambda(\alpha, b, p)$  ( $k \geq 2$ ,  $b \neq 0$  is any complex number,  $0 \leq \alpha < p$  and  $|\lambda| < \pi/2$ ) if there exists  $\delta > 0$  such that

$$\int_0^{2\pi} \operatorname{Re} \left\{ 1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \right\} d\theta = 2p\pi \quad (1 - \delta < r < 1) \quad (1.1)$$

and

$$\limsup_{r \rightarrow 1^-} \int_0^{2\pi} \left| \frac{\operatorname{Re}\{e^{i\lambda} [p + \frac{1}{b} (1 + \frac{zf''(z)}{f'(z)} - p)] - \alpha \cos \lambda\}}{p - \alpha} \right| d\theta \leq k\pi \cos \lambda. \quad (1.2)$$

condition (1.1) implies that  $f$  has  $(p-1)$  critical points in  $U$ .

It is noticed that, by giving specific values to  $k, \alpha, b, p$  and  $\lambda$  in  $V_k^\lambda(\alpha, b, p)$ , we obtain the following important subclasses studied by various authors in earlier papers:

(1)  $V_k^\lambda(o, 1, p) = V_k^\lambda(p)$ , is the class of  $p$ -valent functions investigated by Silvia [1].

(2)  $V_k^o(o, 1, 1) = V_k$ , is the class of bounded boundary rotation introduced by Löwner [2] and Paatero [3, 4]  $V_k^o(\alpha, 1, 1) = V_k(\alpha)$ ,  $0 \leq \alpha < 1$ , is the class of functions  $f(z) \in A_1$  studied by Padmanabhan and Parvatham [5],  $V_k^\lambda(o, 1, 1) = V_k^\lambda$ , is the class of functions  $f(z) \in A_1$  investigated by Moulis [6] and Silvia [7],  $V_k^\lambda(\alpha, 1, 1) = V_k^\lambda(\alpha)$ ,  $0 \leq \alpha < 1$ , is the class of functions  $f(z) \in A_1$  investigated by Moulis [8],  $V_k^o(o, b, 1) = V_k(b)$ , is the class of functions  $f(z) \in A_1$  introduced by Nasr [9] and  $V_k^\lambda(\alpha, b, 1) = V_k^\lambda(\alpha, b)$ , is the class of functions  $f(z) \in A_1$  investigated by Lakshma [10, 11].

(3)  $V_2^o(o, 1, p) = C(p)$ , is the class of  $p$ -valent convex functions considered by Goodman [12],  $V_2^o(\alpha, 1, p) = C(\alpha, p)$ , is the class of  $p$ -valent convex functions of order  $\alpha$ ,  $0 \leq \alpha < p$ ,  $V_2^\lambda(o, 1, p) = C^\lambda(p)$ , is the class of functions  $f(z) \in A_p$  for which  $zf'(z)/p$  is  $\lambda$ -spirallike in  $U$ ,  $V_2^\lambda(\alpha, 1, p) = C^\lambda(\alpha, p)$ , is the class of functions  $f(z) \in A_p$  for which  $zf'(z)/p$  is  $\lambda$ -spirallike of order  $\alpha$ ,  $0 \leq \alpha < p$ ,  $V_2^o(o, b, p) = C(b, p)$ , is the class of  $p$ -valent functions satisfying

$$\operatorname{Re} \left[ p + \frac{1}{b} \left( 1 + \frac{zf''(z)}{f'(z)} - p \right) \right] > 0, z \in U.$$

The class  $C(b, p)$  introduced by Aouf [13] and finally for  $k=2$  any function  $f(z) \in V_2^\lambda(\alpha, b, p) = C^\lambda(\alpha, b, p)$  if and only if

$$\operatorname{Re} e^{i\lambda} \left[ p + \frac{1}{b} \left( 1 + \frac{zf''(z)}{f'(z)} - p \right) \right] > \alpha \cos \lambda, z \in U.$$

Also from  $V_k^\lambda(\alpha, b, p)$ , we can obtain the following important subclasses:

(4)  $V_k^o(\alpha, 1, p) = V_k(\alpha, p)$ , is the class of  $p$ -valent functions  $f(z) \in A_p$  satisfying

$$\lim_{r \rightarrow 1^-} \sup \int_0^{2\pi} \left| \frac{\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} - \alpha}{p - \alpha} \right| d\theta \leq k\pi,$$

i.e.,  $f(z) \in V_k(\alpha, p)$  if and only if

$$f'(z) = pz^{p-1} \left\{ \exp \left[ -(p-\alpha) \int_0^{2\pi} \log(1 - ze^{-it}) dm(t) \right] \right\}, m(t) \in M_k.$$

(5)  $V_k^\lambda(\alpha, 1, p) = V_k^0(\alpha, \cos e^{-i\lambda}, p) = V_k^\lambda(\alpha, p)$ ,  $|\lambda| < \frac{\pi}{2}$ , is the class of p-valent functions  $f(z) \in A_p$  satisfying

$$\lim_{r \rightarrow 1^-} \sup \int_0^{2\pi} \left| \frac{\operatorname{Re} \left\{ e^{i\lambda} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} - \alpha \cos \lambda}{p - \alpha} \right| d\theta \leq k\pi \cos \lambda$$

i.e.,  $f(z) \in V_k^\lambda(\alpha, p)$  if and only if

$$f'(z) = pz^{p-1} \exp \left\{ -(p-\alpha) \cos \lambda e^{-i\lambda} \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \right\}, m(t) \in M_k.$$

(6)  $V_k^0(o, b, p) = V_k(b, p)$ , is the class of p-valent functions  $f(z) \in A_p$  satisfying

$$\lim_{r \rightarrow 1^-} \sup \int_0^{2\pi} \left| \operatorname{Re} \left\{ p + \frac{1}{b} \left( 1 + \frac{zf''(z)}{f'(z)} - p \right) \right\} \right| d\theta \leq k\pi p,$$

i.e.,  $f(z) \in V_k(b, p)$  if and only if

$$f'(z) = pz^{p-1} \exp \left\{ -pb \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \right\}, m(t) \in M_k.$$

(7)  $V_k^0(\alpha, b, p) = V_k(\alpha, b, p)$ , is the class of p-valent functions  $f(z) \in A_p$  satisfying

$$\lim_{r \rightarrow 1^-} \sup \int_0^{2\pi} \left| \frac{\operatorname{Re} \left\{ p + \frac{1}{b} \left( 1 + \frac{zf''(z)}{f'(z)} - p \right) \right\} - \alpha}{p - \alpha} \right| d\theta \leq k\pi,$$

i.e.,  $f(z) \in V_k(\alpha, b, p)$  if and only if

$$f'(z) = pz^{p-1} \exp \left\{ -(p-\alpha)b \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \right\}, m(t) \in M_k.$$

We state below some lemmas that are needed in our investigation.

LEMMA 1 [10].  $h(z) \in V_k^\lambda(\alpha, b)$  if and only if

$$(1) h'(z) = \exp \left\{ -(1-\alpha)b \cos \lambda e^{-i\lambda} \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \right\}, m(t) \in M_k.$$

$$(2) h'(z) = [f_1'(z)]^{(1-\alpha)b \cos \lambda e^{-i\lambda}}, f_1 \in V_k.$$

$$(3) h'(z) = [f_2'(z)]^b \cos \lambda e^{-i\lambda}, f_2 \in V_k(\alpha).$$

$$(4) h'(z) = [f_3'(z)]^{(1-\alpha)b}, f_3 \in V_k^\lambda.$$

$$(5) h'(z) = [f_4'(z)]^b, f_4 \in V_k^\lambda(\alpha).$$

$$(6) h'(z) = [f_5'(z)]^{(1-\alpha) \cos \lambda e^{-i\lambda}}, f_5 \in V_k(b).$$

(7) there exists two normalized starlike functions  $s_1(z)$  and  $s_2(z)$  such that

$$f'(z) = \left\{ \frac{\frac{s_1(z)}{z} \frac{k+2}{4} (1-\alpha)b \cos \lambda e^{-i\lambda}}{\frac{s_2(z)}{z} \frac{k-2}{4}} \right\}$$

LEMMA 2[10]. If  $h(z)$  belongs to  $V_k^\lambda(\alpha, b)$ , then  $H(z)$  is defined by

$$H'(z) = \frac{h'(\frac{z+a}{1+\bar{a}z})}{h'(a)(1+\bar{a}z)^2(1-\alpha)b \cos \lambda e^{-i\lambda}}$$

$H(0) = 0, |z| < 1, |a| < 1$  is also in the class  $V_k^\lambda(\alpha, b)$ .

LEMMA 3[10]. Suppose  $h(z) = z + b_2z^2 + b_3z^3 + \dots$  belongs to  $V_k^\lambda(\alpha, b)$  then

$$|b_2| \leq \frac{k}{2}(1-\alpha) |b| \cos \lambda,$$

and

$$|b_3| \leq \frac{1}{3}(1-\alpha) |b| \cos \lambda [1-(1-\alpha) |b| \cos \lambda \frac{k^2}{2}].$$

These bounds are sharp, with equality for the function  $h(z) \in V_k^\lambda(\alpha, b)$  defined by

$$h'(z) = \left[ \frac{(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}} \right]^{(1-\alpha)b \cos \lambda e^{-i\lambda}}.$$

LEMMA 4[1].  $g(z) \in V_k^\lambda(p), p \geq 1$ , if and only if  $g'(z) = pz^{p-1}[f_3'(z)]^p$  for some  $f_3 \in V_k^\lambda$ .

2. REPRESENTATION FORMULAS FOR THE CLASS  $V_k^\lambda(\alpha, b, p)$ .

LEMMA 5.  $f(z) \in V_k^\lambda(\alpha, b, p)$  if and only if

$$f'(z) = pz^{p-1} [h'(z)]^{\frac{(p-\alpha)}{1-\alpha}} = pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1} \tag{2.1}$$

for some  $h \in V_k^\lambda(\alpha, b)$ .

PROOF. Let  $f'(z) = pz^{p-1} [h'(z)]^{\frac{(p-\alpha)}{1-\alpha}} = pz^{p-1} + \sum_{n=p+1}^{\infty} n a_n z^{n-1}$  for

$$h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k^\lambda(\alpha, b), z \in U.$$

By direct computation, we obtain

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} \{ e^{i\lambda} [p + \frac{1}{b} (1 + \frac{zf''(z)}{f'(z)} - p)] - \alpha \cos \lambda \}}{p-\alpha} \right| d\theta =$$

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} \{ e^{i\lambda} [1 + \frac{1}{b} \frac{zh''(z)}{h'(z)}] - \alpha \cos \lambda \}}{1-\alpha} \right| d\theta$$

and the result follows from (1.2).

An immediate consequence of Lemmas 5 and 1 is

THEOREM 1.  $f(z) \in V_k^\lambda(\alpha, b, p)$  if and only if

$$(1) f'(z) = pz^{p-1} \exp \{ -(p-\alpha)b \cos \lambda e^{-i\lambda} \int_0^{2\pi} \log(1-ze^{-it}) dm(t) \}, m(t) \in M_k.$$

$$(2) \quad f'(z) = pz^{p-1} [f'_1(z)]^{(p-\alpha)b \cos \lambda} e^{-i\lambda}, \quad f_1 \in V_k.$$

$$(3) \quad f'(z) = pz^{p-1} [f'_2(z)]^{\left(\frac{p-\alpha}{1-\alpha}\right)b \cos \lambda} e^{-i\lambda}, \quad f_2 \in V_k(\alpha).$$

$$(4) \quad f'(z) = pz^{p-1} [f'_3(z)]^{(p-\alpha)b}, \quad f_3 \in V_k^\lambda.$$

$$(5) \quad f'(z) = pz^{p-1} [f'_4(z)]^{\left(\frac{p-\alpha}{1-\alpha}\right)b}, \quad f_4 \in V_k(\alpha)$$

$$(6) \quad f'(z) = pz^{p-1} [f'_5(z)]^{(p-\alpha)\cos \lambda} e^{-i\lambda}, \quad f_5 \in V_k(b).$$

(7) there exists two normalized starlike functions  $s_1(z)$  and  $s_2(z)$  such that

$$f'(z) = pz^{p-1} \frac{\left\{ \frac{s_1(z)}{z} \right\}^{\frac{k+2}{4}} (p-\alpha)b \cos \lambda e^{-i\lambda}}{\left\{ \frac{s_2(z)}{z} \right\}^{\frac{k-2}{4}}}$$

Also an immediate consequence of Theorem 1 (4) and Lemma 4 is

THEOREM 2.  $f(z) \in V_k^\lambda(\alpha, b, p)$  if and only if

$$f'(z) = pz^{p-1} \left[ \frac{g'(z)}{pz^{p-1}} \right]^{\left(\frac{p-\alpha}{p}\right)b}, \quad g \in V_k^\lambda(p)$$

### 3. COEFFICIENT ESTIMATES FOR THE CLASS $V_k^\lambda(\alpha, b, p)$ .

THEOREM 3. If  $f(z) = z^p + \sum_{n=p+1}^\infty a_n z^n \in V_k^\lambda(\alpha, b, p)$ , then

$$|a_{p+1}| \leq \left[ \frac{p(p-\alpha)}{p+1} \right] k |b| \cos \lambda, \tag{3.1}$$

$$|a_{p+2}| \leq p \left( \frac{p-\alpha}{p+2} \right) |b| \cos \lambda \left\{ [1 - (1-\alpha)|b| \cos \lambda \frac{k^2}{2}] + (p-1) |b| \cos \lambda \frac{k^2}{2} \right\} \tag{3.2}$$

These bounds are sharp with equality for

$$f'_*(z) = pz^{p-1} \left[ \frac{\left(\frac{1-z}{k}\right)^{\frac{k}{2}-1} (p-\alpha)b \cos \lambda e^{-i\lambda}}{(1+z)^{\frac{k}{2}+1}} \right] \tag{3.3}$$

PROOF. By Lemma 5, there exists an  $h(z) = z + \sum_{n=2}^\infty b_n z^n \in V_k^\lambda(\alpha, b)$  such that

$$f'(z) = pz^{p-1} + \sum_{n=p+1}^\infty n a_n z^{n-1} = pz^{p-1} \left[ 1 + \sum_{n=2}^\infty n b_n z^{n-1} \right]^{\left(\frac{p-\alpha}{1-\alpha}\right)} \tag{3.4}$$

Expanding the right side of (3.4), we obtain

$$f'(z) = pz^{p-1} + 2p \left( \frac{p-\alpha}{1-\alpha} \right) b_2 z^p + p \left\{ 3 \left( \frac{p-\alpha}{1-\alpha} \right) b_3 + 2 \left( \frac{p-\alpha}{1-\alpha} \right) \left( \frac{p-1}{1-\alpha} \right) b_2^2 \right\} z^{p+1} + \dots \tag{3.5}$$

Equating coefficients from (3.5) and (3.4), we have

$$(p+1)a_{p+1} = 2p\left(\frac{p-\alpha}{1-\alpha}\right)b_2,$$

$$(p+2)a_{p+2} = p\left\{3\left(\frac{p-\alpha}{1-\alpha}\right)b_3 + 2\left(\frac{p-\alpha}{1-\alpha}\right)\left(\frac{p-1}{1-\alpha}\right)b_2^2\right\}.$$

Thus the result follows from lemma 3.

#### 4. SHARP RADIUS OF CONVEXITY OF THE CLASS $V_k^\lambda(\alpha, b, p)$ .

LEMMA 6. If  $f \in V_k^\lambda(\alpha, b, p)$ , then the transformation  $F_a$  satisfying

$$F'_a(z) = \frac{p a^{p-1} z^{p-1} f'\left(\frac{z+a}{1+\bar{a}z}\right)}{f'(a)(z+a)^{p-1}(1+\bar{a}z)^{2(p-\alpha)}b \cos \lambda e^{-i\lambda} - p+1}, \quad z \in U \quad (4.1)$$

and  $F_a(0) = 0$ , is in  $V_k^\lambda(\alpha, b, p)$  for all  $|a| < 1$ .

The proof of this lemma follows by using lemmas 5 and 2.

LEMMA 7. For  $|z| \leq r$  and  $f$  ranging over  $V_k^\lambda(\alpha, b, p)$ , the domain of values of  $\frac{zf''(z)}{f'(z)}$  is the disc with center

$$\left( \frac{[2(p-\alpha)\operatorname{Re}\{b \cos \lambda e^{-i\lambda}\} - p+1] r^2 + (p-1)}{1-r^2}, \frac{2(p-\alpha)\operatorname{Im}\{b \cos \lambda e^{-i\lambda}\} r^2}{1-r^2} \right)$$

and radius  $\frac{p(p-\alpha)k|b| \cos \lambda r}{1-r^2}$ .

PROOF. Whenever

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in V_k^\lambda(\alpha, b, p), \quad \lim_{z \rightarrow 0} \left\{ \frac{f''(z) - p(p-1)z^{p-2}}{z^{p-1}} \right\} = p(p-1)a_{p+1}.$$

For  $f \in V_k^\lambda(\alpha, b, p)$ , let  $F_a(z) = z^p + \sum_{n=p+1}^{\infty} A_n z^n \in V_k^\lambda(\alpha, b, p)$  be given by Lemma 6 for  $|a| < 1$ . By direct calculation we have

$$p(p+1)A_{p+1} = p(1-|a|^2) \frac{f''(a)}{f'(a)} - \frac{p[2(p-\alpha)b \cos \lambda e^{-i\lambda} - p+1]|a|^2 + p(p-1)}{a}. \quad (4.2)$$

Combining (3.1) and (4.2), we obtain

$$\left| \frac{f''(a)}{f'(a)} - \frac{[2(p-\alpha)b \cos \lambda e^{-i\lambda} - p+1]|a|^2 + p(p-1)}{a(1-|a|^2)} \right| \leq \frac{p(p-\alpha)k|b| \cos \lambda}{(1-|a|^2)} \quad (4.3)$$

From (4.3) it follows that for  $|z| = r < 1$ ,

$$\left| \frac{zf''(z)}{f'(z)} - \frac{[2(p-\alpha)b \cos \lambda e^{-i\lambda} - p+1]r^2 + p(p-1)}{(1-r^2)} \right| \leq \frac{p(p-\alpha)k|b| \cos \lambda r}{(1-r^2)} \quad (4.4)$$

and the proof is completed.

THEOREM 4. The sharp radius of convexity of the class  $V_k^\lambda(\alpha, b, p)$  is

$$r_0 = 2\{(p-\alpha)k|b| \cos \lambda + [(p-\alpha)^2 k^2 |b|^2 \cos^2 \lambda - 4(2(1-\frac{\alpha}{p})\operatorname{Re}\{b \cos \lambda e^{-i\lambda}\} - 1)]^{\frac{1}{2}}\}^{-1} \quad (4.5)$$

PROOF. From (4.4), we have

$$\left| \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{-p} \left( \frac{1 + (2(1 - \frac{\alpha}{p})b \cos \lambda e^{-i\lambda} - 1)r^2}{1-r^2} \right) \right| \leq \frac{p(p-\alpha)k|b|\cos \lambda r}{1-r^2}$$

which implies that

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq p \left[ \frac{1 - (p-\alpha)k|b|\cos \lambda r + (2(1 - \frac{\alpha}{p}) \operatorname{Re}\{b \cos \lambda e^{-i\lambda} - 1\})r^2}{1-r^2} \right].$$

Therefore  $\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0$  for  $|z| < r_0$  is given by (4.5). The function  $f_x(z)$  defined by

$$f'_x(z) = pz^{p-1} \left[ \frac{\frac{k}{2} - 1 + (p-\alpha)b \cos \lambda e^{-i\lambda}}{(1-\varepsilon z)^{\frac{k}{2} + 1}} \right] \tag{4.6}$$

when  $z = r$ ,  $e = \frac{r+p e^{i\lambda} \sqrt{\frac{b}{1-b}}}{1+rpe^{i\lambda} \sqrt{\frac{b}{1-b}}}$  and  $\varepsilon = \frac{r-pe^{i\lambda} \sqrt{\frac{b}{1-b}}}{1-rpe^{i\lambda} \sqrt{\frac{b}{1-b}}}$

shows that the bound in (4.5) is sharp.

5. DISTORTION AND ROTATION THEOREMS FOR THE CLASS  $V_k^\lambda(\alpha, b, p)$ .

In [11] Lakshma used variational method to solve the extremal problems for the class  $V_k^\lambda(\alpha, b)$ . He proved the following:

LEMMA 8[11]. Let  $\xi \neq 0$  be a given point in  $U$  and let  $H(x_1, x_2, \dots, x_{n+1})$  be analytic in a neighbourhood of each point  $H(h'(\xi), h''(\xi), \dots, h^{(n)}(\xi), \xi)$ ,  $h \in V_k^\lambda(\alpha, b)$ .

Then the functional

$$J(h') = \operatorname{Re} H(h'(\xi), h''(\xi), \dots, h^{(n)}(\xi), \xi)$$

attains its maximum (or minimum) in  $V_k^\lambda(\alpha, b)$  only for a function of the form

$$h'(z) = \prod_{j=1}^M (1-\varepsilon_j z)^{\beta_j} (1-\alpha)b \cos \lambda e^{-i\lambda} \prod_{j=1}^N (1-e_j z)^{-\alpha_j} (1-\alpha)b \cos \lambda e^{-i\lambda} \tag{5.1}$$

where  $M \leq n$ ,  $N \leq n$ ,  $|\varepsilon_j| = |e_j| = 1$ ,  $\sum_{j=1}^M \beta_j \leq \frac{k}{2} - 1$ ,  $\sum_{j=1}^N \alpha_j \leq \frac{k}{2} + 1$ .

LEMMA 9[11]. If  $h(z) \in V_k^\lambda(\alpha, b)$ , then we have

$$|e|^{\min} |\varepsilon| = 1 \left| \frac{(1-\varepsilon z)^{\frac{k}{2} - 1} A}{(1-\varepsilon z)^{\frac{k}{2} + 1} A} \right| \leq |h'(z)| \leq |e|^{\max} |\varepsilon| = 1 \left| \frac{(1-\varepsilon z)^{\frac{k}{2} - 1} A}{(1-\varepsilon z)^{\frac{k}{2} + 1} A} \right|,$$

where  $A = (1-\alpha)b \cos \lambda e^{-i\lambda}$ .

Bounds are sharp with properly selected  $e$  and  $\varepsilon$  with  $M = N = 1$  in (5.1).

LEMMA 10[11]. If  $h(z) \in V_k^\lambda(\alpha, b)$ , then for  $|z| = r < 1$ , we obtain

$$|\arg h'(z)| \leq (1-\alpha)k|b|\cos \lambda \arcsin |z|.$$

The proof of the following two theorems follows from Lemma 5 and the above bounds.

THEOREM 5. If  $f \in V_k^\lambda(\alpha, b, p)$ , then for  $|z| = r < 1$ , we obtain

$$|e|^{\min} |\varepsilon| = 1 \left| \frac{\left(\frac{k}{2} - 1\right)B}{(1-\varepsilon z)} \frac{(1-\varepsilon z)}{\left(\frac{k}{2} + 1\right)B} \right| \leq \left| \frac{f'(z)}{pz^{p-1}} \right| \leq |e|^{\max} |\varepsilon| = 1 \left| \frac{\left(\frac{k}{2} - 1\right)B}{(1-\varepsilon z)} \frac{(1-\varepsilon z)}{\left(\frac{k}{2} + 1\right)B} \right| ,$$

where  $B = (p-\alpha)b \cos \lambda e^{-i\lambda}$ .

Bounds are sharp with equality for  $f'(z) = pz^{p-1} [h'(z)]^{\frac{(p-\alpha)}{1-\alpha}}$ , where  $h(z)$  is defined by (5.1) with properly selected  $e$  and  $\varepsilon$  with  $M = N = 1$ .

THEOREM 6. If  $f \in V_k^\lambda(\alpha, b, p)$ , then for  $|z| = r < 1$ , we obtain

$$|\arg f'(z)| \leq (p-1)^\theta + (p-\alpha)k |b| \cos \lambda \arcsin |z|.$$

6. HARDY CLASSES FOR THE CLASS  $V_k^\lambda(\alpha, b)$ .

In order to obtain the  $H^\mu$  classes for the class  $V_k^\lambda(\alpha, b)$ , we will use the following lemmas.

LEMMA 11[7]. IF  $f_3 \in V_k^\lambda$ . Then for  $|z|=r$ ,  $|\arg f_3'(re^{i\theta})| \leq k \cos \lambda \arcsin r$ .

LEMMA 12[14]. If  $f' \in H^\mu (0 < \mu < 1)$  then  $f \in H^{1-\mu}$  where, for  $\mu=1$ ,  $H^\infty$  is the class of bounded functions.

LEMMA 13[15]. If  $f(z) \in H^\mu (0 < \mu < 1)$  and  $f(z) = \sum_{n=0}^\infty a_n z^n$ , then  $a_n = o(n^{\frac{1}{\mu} - 1})$ .

LEMMA 14[1]. If  $f_3 \in V_k^\lambda$ , then  $f_3' \in H^\mu$  for all  $\mu < \frac{2}{\cos^2 \lambda (k+2)}$  and  $f_3 \in H^\eta$  for  $\eta < \frac{2}{[\cos^2 \lambda (k+2) - 2]}$ ,  $\cos^2 \lambda > \frac{2}{(k+2)}$ . Furthermore, if  $f_3'$  is not of the form

$$f_3'(z) = [f_0'(z)]^{\cos \lambda} e^{-i\lambda} \text{ where } f_0 \text{ is given by}$$

$$f_0'(z) = (1-ze^{-it})^{-\left(\frac{k}{2}+1\right)} \exp \left\{ \int_0^{2\pi} -\log(1-ze^{-it}) dm(t) \right\} \tag{6.1}$$

$(m(t))$  a probability measure on  $[0, 2\pi]$ , then there exists  $\delta = \delta(f_3) > 0$  and  $\varepsilon = \varepsilon(f_3) > 0$  such that

$$f_3'(z) \in H^{\cos^2 \lambda (k+2)} \text{ and } f_3 \in H^{\frac{(2+\varepsilon)}{[\cos^2 \lambda (k+2) - 2]}} \text{ for } \cos^2 \lambda > \frac{2}{(k+2)} .$$

THEOREM 7. If  $h \in V_k^\lambda(\alpha, b)$ ,  $\text{Re}\{b\} > 0$ , then  $h' \in H^\mu$  for all

$$\mu < \frac{2}{(1-\alpha)\text{Re}\{b\} \cos^2 \lambda (k+2)} \text{ and } h \in H^\eta \text{ for } \eta < \frac{2}{[(1-\alpha)\text{Re}\{b\} \cos^2 \lambda (k+2) - 2]} ,$$

where  $\cos^2\lambda > \frac{2}{(1-\alpha)\operatorname{Re}\{b\}(k+2)}$ . Furthermore, if  $h'$  is not of the form  $h'(z) =$

$[f_3'(z)]^{(1-\alpha)b} = [f_0'(z)]^{(1-\alpha)b} \cos\lambda e^{-i\lambda}$  where  $f_0(z)$  is given by (6.1), then there exists  $\varepsilon = \varepsilon(h) > 0$  and  $\delta = \delta(h) > 0$  such that

$$h' \in H^{\varepsilon+} \frac{2}{[(1-\alpha)\operatorname{Re}\{b\}\cos^2\lambda(k+2)]} \quad \text{and} \quad h \in H^{\delta+} \frac{2}{[(1-\alpha)\operatorname{Re}\{b\}\cos^2\lambda(k+2)-2]}$$

for  $\cos^2\lambda > \frac{2}{(1-\alpha)\operatorname{Re}\{b\}(k+2)}$ .

PROOF. If  $h \in V_k^\lambda(\alpha, b)$ , then it follows from Lemma 1(4) that

$$h'(z) = [f_3'(z)]^{(1-\alpha)b}, \quad f_3 \in V_k^\lambda.$$

Then

$$|h'(z)|^\mu = |f_3'(z)|^\mu (1-\alpha)\operatorname{Re}\{b\} \cdot \exp\{-\mu(1-\alpha)\operatorname{Im}\{b\}\arg f_3'(z)\}.$$

By Lemma 11, the exponential factor is bounded. Thus the result follows from Lemmas 12 and 14. From Theorem 7 and Lemma 13, we obtain a growth estimate for the Taylor's coefficients of  $h \in V_k^\lambda(\alpha, b)$ .

COROLLARY 1. If  $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_k^\lambda(\alpha, b)$  and  $\cos^2\lambda > \frac{2}{(1-\alpha)\operatorname{Re}\{b\}(k+2)}$ , then

$$b_n = o\left(n^{\frac{[(1-\alpha)\operatorname{Re}\{b\}\cos^2\lambda(k+2)-4]}{2}}\right).$$

Acknowledgment. The author is thankful to Professor Dr. S. M. Shah for reading the manuscript and for helpful suggestions.

REFERENCES

1. SILVIA, E.M., A note of special classes of p-valent functions, Rocky Mt. J. Math. 9(2)(1979), 365-370.
2. LOWNER, K., Untersuchungen über die Verzerrung bei konformen Abbildungen Einheitskreises  $|z| < 1$ , die durch Funktionen mit nicht verschwindender Ableitung geliefert werden, Ber. Königl. Sachs. Ges. Wiss. Leipzig 69 (1917), 89-106.
3. PAATERO, V., Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind, Ann. Acad. Sci. Fenn. Ser. A. (33) 9 (1931), 1-77.
4. PAATERO, V., Über Gebiete von beschränkter Raddrehung, Ann. Acad. Sci. Fenn. Ser. A. 37 (1933), 1-20.
5. PADMANBHAN, K.S. and PARVATHAM, R., Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math. 31 (3) (1975), 311-323.
6. MOULIS, E.J. Jr., A generalization of univalent functions with bounded boundary rotation, Trans. Amer. Math. Soc., 174(1972), 369-381.
7. SILVIA, E.M., A variational method on certain classes of functions, Rev. Roum. Math. Pures et Appl., 21(5) (1979), 549-557.
8. MOULIS, E.J. Jr., Generalizations of the Robertson functions, Pacific J. Math. 81(1) (1979), 167-174.
9. NASR, M.A., On a class of functions with bounded boundary rotation, Bull. of the Institute of Math., Academia Sinica 5(1977), 27-36.

10. LAKSHMA, R., On certain classes of functions with bounded boundary rotation, Indian J. Pure and Appl. Math., 13(2) (1982), 195-204.
11. LAKSHMA, R., A variational method on certain classes of analytic functions, Indian J. Pure and Appl. Math., 14(5) (1983), 601-613.
12. GOODMAN, A.W., On the SchwarzpChristoffel transformation and p-valent functions, Trans. Amer. Math. Soc., 68(1950), 204-223.
13. AOUF, M.K., p-Valent classes related to convex functions of complex order, to appear, Rocky Mt. J. Math.
14. HARDY, G.H., and LITTLEWOOD, J.E., Some properties of fractional integrals II, Math. Z., 34(1932), 403-439.
15. DUREN, P.L., Theory of  $H^p$  spaces, Pure and Appl. Math. 38 , Academic Press, New York, 1970.

## Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

### Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from "Qualitative Theory of Differential Equations," allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

|                        |                  |
|------------------------|------------------|
| Manuscript Due         | February 1, 2009 |
| First Round of Reviews | May 1, 2009      |
| Publication Date       | August 1, 2009   |

### Guest Editors

**José Roberto Castilho Piqueira**, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; [piqueira@lac.usp.br](mailto:piqueira@lac.usp.br)

**Elbert E. Neher Macau**, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil ; [elbert@lac.inpe.br](mailto:elbert@lac.inpe.br)

**Celso Grebogi**, Department of Physics, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; [grebogi@abdn.ac.uk](mailto:grebogi@abdn.ac.uk)