

## ON SOME SPACES OF SUMMABLE SEQUENCES AND THEIR DUALS

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ABSTRACT. Suppose that  $S$  is the space of all summable sequences  $\alpha$  with  $\|\alpha\|_S = \sup_{n \geq 0} \left| \sum_{j=n}^{\infty} \alpha_j \right|$  and  $J$  the space of all sequences  $\beta$  of bounded variation with  $\|\beta\|_J = \left| \beta_0 \right| + \sum_{j=1}^{\infty} \left| \beta_j - \beta_{j-1} \right|$ . Then for  $\alpha$  in  $S$  and  $\beta$  in  $J$   $\left| \sum_{j=0}^{\infty} \alpha_j \beta_j \right| \leq \|\alpha\|_S \|\beta\|_J$ ; this inequality leads to the description of the dual space of  $S$  as  $J$ . It, related inequalities, and their consequences are the content of this paper. In particular, the inequality cited above leads directly to the Stolz form of Abel's theorem and provides a very simple argument. Also, some other sequence spaces are discussed.

KEY WORDS AND PHRASES. Dual spaces, sequence spaces, bounded operators.

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### 1. INTRODUCTION.

Suppose  $S$  is the space of all summable sequences  $s = (s_n)_{n=0}^{\infty}$ , with norm given by  $\|s\|_S = \sup_{n \geq 0} \left| \sum_{j=n}^{\infty} s_j \right|$ , the remainder in the sum of  $s$  after  $n$ -terms. An example of such an  $\bar{s}$ , summable but not absolutely summable, is given by  $s_n = \frac{(-1)^n}{n+1}$ ,  $n = 0, 1, 2, 3, \dots$ , so that  $\sum_{j=0}^{\infty} \frac{(-1)^j}{j+1}$  is an alternating convergent series. Here, as is readily verified,  $\|s\|_S = \sup_{n \geq 0} \left| \sum_{j=n}^{\infty} \frac{(-1)^j}{j+1} \right| = \ln 2$ , the natural log of 2. This paper is based on the observation that  $S$ , with this norm, is naturally isomorphic to  $c_0$ , the space of all sequences having limit zero endowed with the supremum norm. Moreover, if we consider  $S$  on its own, we get several interesting results. For example, since  $S$  is isomorphic to  $c_0$  and  $c_0^*$ , the dual of  $c_0$ , is isomorphic to  $\ell_1$ , the space of all absolutely summable sequences, then it follows that  $S^*$ , the dual of  $S$ , is isomorphic to  $\ell_1$ .

However, if we compute the dual of  $S$  without using the isomorphism with  $c_0$ , we find that  $S^*$  is isomorphic to  $J$ , the space of all sequences  $\beta$  of bounded variation with

$$\|\beta\|_J = |\beta_0| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}|; \text{ an example of such a } \beta \text{ being given}$$

$$\text{by } \beta_n = \frac{1}{n+1} \quad n = 0, 1, 2, 3, \dots \text{ so that } \|\beta\|_J = 1 + \sum_{n=1}^{\infty} \left[ \frac{1}{n} - \frac{1}{n+1} \right] = 2.$$

Of course,  $J$  is isomorphic to  $\ell_1$ . Another example is that we get the following

new inequality: If  $\beta \in J$  and  $s \in S$   $\left| \sum_{j=0}^{\infty} s_j \beta_j \right| < \|s\|_S \cdot \|\beta\|_J$ , which can be used to give a proof of Abel's Continuity Theorem.

Similarly, we consider the space  $H$  of all sequences  $\lambda = (\lambda_n)_{n=0}^{\infty}$  such that

$$\|\lambda\|_H = \sup_{n>0} \left| \sum_{j=0}^n \lambda_j \right| \text{ is finite; for example } \lambda_n = (-1)^n, \quad n = 0, 1, 2, 3, \dots, \text{ so}$$

that  $\|\lambda\|_H = 1$ . We get the following inequality: If  $\lambda \in H$ ,  $\beta \in J$ , and  $\rho = \lim \beta$

then  $\left| \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) \right| \leq \|\lambda\|_H \cdot \|\beta\|_J$ . Consequently we have that  $\phi \in J^*$  if

and only if  $\phi(\beta) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1})$  for  $\beta \in J$ .

Also, we give characterizations of the bounded linear operators on  $S$  easily obtained from a description of operators on  $c_0$ . These results, however, may be obtained using the uniform boundedness principle.

To make the presentation reasonably self-contained, we shall include a resume of pertinent results and definitions.

## 2. PRELIMINARIES

DEFINITION 2.1. Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two sequence spaces, which are Banach spaces with the respective norms. For a fixed  $y = (y_n) \in Y$  define the mapping  $\phi_y: X \rightarrow \mathbb{R}$  by  $\phi_y(x) = \sum_{n=0}^{\infty} x_n y_n$  for those  $x = (x_n)$  in  $X$  for which the series converges.

Assume that  $|\phi_y(x)| \leq M \|x\|_X$  with  $M$  some absolute constant, that is,  $\phi_y$  is a bounded linear functional on  $X$  for any fixed  $y \in Y$ . Let  $X^*$  be the space of all bounded linear functionals  $\phi$  on  $X$ , endowed with norm

$$\|\phi\|_{X^*} = \sup_{\|x\|_X=1} |\phi(x)| \text{ and } \psi \text{ the mapping from } Y \text{ to } X^* \text{ given by}$$

$\psi(y) = \phi_y$ . If  $\psi$  is onto and  $\|y\|_Y = \|\phi_y\|_{X^*}$ , then we say that  $Y$  is the dual of  $X$ , in this sense we write  $Y = X^*$ .

DEFINITION 2.2. We define the sequence spaces  $S$ ,  $\ell_1$ ,  $\ell_{\infty}$ ,  $c_0$ , and  $H$  respectively by

$$S = \{s = (s_n)_{n=0}^{\infty}, \text{ summable sequences such that } \|s\|_S = \sup_{p \geq 0} \left| \sum_{j=p}^{\infty} s_j \right|\},$$

$$\ell_1 = \{\delta = (\delta_n)_{n=0}^{\infty}, \|\delta\|_{\ell_1} = \sum_{j=0}^{\infty} |\delta_j| < \infty\},$$

$$\ell_{\infty} = \{\mu = (\mu_n)_{n=0}^{\infty}, \|\mu\|_{\ell_{\infty}} = \sup_{j \geq 0} |\mu_j| < \infty\},$$

$$c_0 = \{\gamma = (\gamma_n)_{n=0}^{\infty}, \lim_{n \rightarrow \infty} \gamma_n = 0, \|\gamma\|_{c_0} = \sup_{n \geq 0} |\gamma_n|\}$$

$$H = \{ \lambda = (\lambda_n)_{n=0}^\infty, \|\lambda\|_H = \sup_{n \geq 0} \left| \sum_{j=0}^n \lambda_j \right| < \infty \}.$$

We believe that this is the first time that  $S$  is taken as a space with this norm.

**THEOREM 2.3.** The space  $S$  is a Banach space, isometrically isomorphic to  $c_0$ . The isometry  $T: S \rightarrow c_0$  being given by  $T(s) = (r_n(s))_{n=0}^\infty$  and  $T^{-1}: c_0 \rightarrow S$  by  $T^{-1}(\gamma) = (\gamma_n - \gamma_{n+1})_{n=0}^\infty$ , where  $r_n(s) = \sum_{j=n}^\infty \alpha_j$ .

**DEFINITION 2.4.** Two Banach spaces  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are isomorphic equivalents if there exist a one-to-one and onto linear mapping  $T: X \rightarrow Y$  such that  $N\|\cdot\|_Y \leq \|Tx\|_Y \leq M\|\cdot\|_X$  with  $N$  and  $M$  absolute constants. If there exists such a  $T$  with  $\|Tx\|_Y = \|\cdot\|_X$ , then we say that  $X$  and  $Y$  are isometrically isomorphic.

Notice in Theorem 2.3  $T$  and  $T^{-1}$  can be represented as infinite matrices,

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 0 & 1 & -1 & 0 \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

**THEOREM 2.5.**  $M$  is a bounded linear operator from  $c_0$  into  $c_0$  if and only if  $M$  is represented by an infinite matrix with columns in  $c_0$  and rows in  $\ell_1$ , uniformly bounded in  $\ell_1$ . That is if  $M = (m_{nk})$  then

- i)  $\lim_{n \rightarrow \infty} m_{nk} = 0$  for any fixed  $k$ ,
- ii)  $\sup_{n \geq 0} \sum_{k=0}^\infty |m_{nk}| < \infty$ .

**THEOREM 2.6.**  $\phi$  is in  $c_0^*$  if and only if there is a  $\delta = (\delta_n) \in \ell_1$  such that for any  $\gamma = (\gamma_n) \in c_0$   $\phi(\gamma) = \sum_{n=0}^\infty \delta_n \gamma_n$ , in which case there is only one such  $\delta = (\delta_n) \in \ell_1$  and  $\|\phi\|_{c_0^*} = \|\delta\|_{\ell_1}$ .

THEOREM 2.7. The space  $J$  is a Banach space, isometrically isomorphic to  $\ell_1$ . The isometry  $L: J \rightarrow \ell_1$  is given by  $L(\beta) = (\beta_n - \beta_{n-1})_{n=0}^{\infty}$ ,

$\beta_{-1} = 0$ , and  $L^{-1}: \ell_1 \rightarrow J$  by  $L^{-1}(\delta) = (\sum_{j=0}^n \delta_j)_{n=0}^{\infty}$  with  $\delta = (\delta_n) \in \ell_1$  and  $\beta = (\beta_n) \in J$ .

THEOREM 2.8. The space  $H$  is a Banach space isometrically isomorphic to  $\ell_{\infty}$ .

The isometry  $\psi: H \rightarrow \ell_{\infty}$  is given by  $\psi(\lambda) = (\sum_{j=0}^n \lambda_j)_{n=0}^{\infty}$  and  $\psi^{-1}: \ell_{\infty} \rightarrow H$

by  $\psi^{-1}(\mu) = (\mu_n - \mu_{n-1})_{n=0}^{\infty}$  with  $\mu_{-1} = 0$ ,  $\mu = (\mu_n) \in \ell_{\infty}$ ,  $\lambda = (\lambda_n) \in H$ .

### 3. THE DUAL OF $S$

THEOREM 3.1 (Holder's type inequality). If  $s = (s_n) \in S$  and  $\beta = (\beta_n) \in J$

then  $|\sum_{n=0}^{\infty} s_n \beta_n| \leq \|s\|_J \cdot \|\beta\|_S$ .

Proof: For  $N \geq 1$ , we notice that

$$\left| \sum_{n=0}^{N-1} s_n \beta_n - r_0(s) \beta_0 - \sum_{n=1}^N r_n(s) [\beta_n - \beta_{n-1}] \right| = |r_N(s) \beta_N| = |r_N(s)| \cdot |\beta_N|,$$

where  $r_n(s) = \sum_{j=n}^{\infty} s_j$ . Now since  $s$  is summable and  $\beta$  is bounded, it

follows that  $|r_N(s)| \rightarrow 0$  as  $N \rightarrow \infty$ . Therefore we have proved that

$$\sum_{n=0}^{\infty} s_n \beta_n = r_0(s) \beta_0 + \sum_{n=1}^{\infty} r_n(s) [\beta_n - \beta_{n-1}]; \text{ consequently}$$

$$\left| \sum_{n=0}^{\infty} s_n \beta_n \right| \leq \sup_{n \geq 0} |r_n(s)| (|\beta_0| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}|),$$

so that the theorem is proved.

Consider the mapping  $\phi_{\beta}: S \rightarrow \mathbb{R}$  defined by  $\phi_{\beta}(s) = \sum_{n=0}^{\infty} s_n \beta_n$  where  $\beta = (\beta_n)$  is a fixed element in  $J$ . In view of Theorem 3.1, we have  $\phi_{\beta}$  is a bounded linear functional on  $S$ . At this point, a natural question is: are these all the bounded linear functionals on  $S$ ? The next result tells us that the answer is yes.

THEOREM 3.2. (Duality Theorem). If  $\phi \in S^*$  then there is a unique  $\beta = (\beta_n) \in J$

such that  $\phi = \phi_{\beta}$ , that is,  $\phi(s) = \sum_{n=0}^{\infty} s_n \beta_n$  for any  $s = (s_n) \in S$ . Moreover

$\|\phi\|_{S^*} = \|\beta\|_J$ . Conversely if  $\phi(s) = \sum_{n=0}^{\infty} s_n \beta_n$  then  $\phi \in S^*$  so that the mapping

$\psi: J \rightarrow S^*$  defined by  $\psi(\beta) = \phi_{\beta}$  is an isometric isomorphism from  $J$  to  $S^*$ .

Proof: If  $\phi(s) = \sum_{n=0}^{\infty} s_n \beta_n$ , then we already have seen by Theorem 3.1 that  $\phi$  is a bounded linear functional on  $S$ , that is  $\phi \in S^*$ . So it remains to prove the first part. Let  $\phi \in S^*$ ; then using Theorem 2.3, we have

$$\phi(s) = \phi(T^{-1} \circ T(s)) = (\phi \circ T^{-1})(T(s)) = (\phi \circ T^{-1})\left(\left(\sum_{j=n}^{\infty} s_j\right)_{n=0}^{\infty}\right).$$

Now notice that  $\phi \circ T^{-1} \in c_0^*$ , so that there is a  $y = (y_n) \in l_1$  (see Theorem 2.6) such that

$$\phi(s) = (\phi \circ T^{-1})\left(\left(\sum_{j=n}^{\infty} s_j\right)_{n=0}^{\infty}\right) = \sum_{n=0}^{\infty} y_n \left(\sum_{j=n}^{\infty} s_j\right) \text{ and } \|\phi \circ T^{-1}\|_{c_0^*} = \|y\|_{l_1}. \quad (3.3)$$

Observe that (3.3) can be written  $\phi(s) = \sum_{n=0}^{\infty} s_n \sum_{j=0}^n y_j$ ; therefore we may let  $\beta = (\beta_n)$

where  $\beta_n = \sum_{j=0}^n y_j$ .

Since  $\beta = L^{-1}(y)$ ,  $\beta$  is in  $J$ . Consequently given  $\phi \in S^*$  there is a  $\beta \in J$  such that  $\phi = \phi_\beta$ . On the other hand Theorem 2.6 tells us that

$$\|\phi \circ T^{-1}\|_{c_0^*} = \sum_{n=0}^{\infty} |y_n|; \text{ therefore } \|\beta\|_J = \sum_{n=0}^{\infty} |y_n| = \|\phi \circ T^{-1}\|_{c_0^*} \leq \|T^{-1}\| \cdot \|\phi\|_{S^*}.$$

Now as  $\|T^{-1}\| \leq 1$ , it follows that  $\|\beta\|_J \leq \|\phi\|_{S^*}$ . Also notice that Theorem 3.1 implies  $\|\phi\|_{S^*} \leq \|\beta\|_J$  (since  $\phi = \phi_\beta$ ). Therefore putting together these last two inequalities we have  $\|\phi\|_{S^*} = \|\beta\|_J$ ; consequently the mapping  $\psi: J \rightarrow S^*$  defined by  $\psi(\beta) = \phi_\beta$  is an isometry, so that the theorem is proved.

We have used Theorem 3.1 to characterize all bounded linear functionals on  $S$ . Now we are going to use this theorem again to give a proof of Abel's continuity theorem which is as follows.

**THEOREM 3.4 (Abel's continuity theorem).** If  $\alpha = (\alpha_n)$  is summable and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n, \text{ then } \lim_{z \rightarrow 1} f(z) = f(1), \text{ where } z \text{ is restricted to approach the point } 1 \text{ in such a way that } |z| < 1 \text{ and } \frac{1-z}{1-|z|} \text{ remains bounded.}$$

**Proof:** First of all, let  $C$  be a positive absolute constant such that

$$\frac{1-|z|}{1-|z|} \leq C \text{ and } |z| < 1. \text{ Notice that } \left| \sum_{p=N}^{\infty} \alpha_p z^p \right| \leq \|(z^p)_{p=N}^{\infty}\|_J \cdot \|(\alpha_p)_{p=N}^{\infty}\|_S$$

for  $N \geq 1$  by the above inequality applied to the sequences  $(\alpha_p)$  and  $(z^p)$ ,

since  $(z^p)_{p=N}^{\infty}$  is in  $J$ ; in fact, since  $|z| < 1$

$$\begin{aligned} \|(z^p)_{p=N}^{\infty}\|_J &= |z|^N + |z^N - z^{N+1}| + |z^{N+1} - z^{N+2}| + \dots = \\ &= |z|^N + |1-z|(|z|^N + |z|^{N+1} + \dots) = |z|^N \left(1 + \frac{1-|z|}{1-|z|}\right). \end{aligned}$$

Now using the hypothesis we have  $\|(z^p)_{p=N}^{\infty}\|_J \leq 1 + C$ . Consequently,

$$\left| \sum_{p=N}^{\infty} \alpha_p z^p \right| \leq (1+C)\|(\alpha_p)_{p=N}^{\infty}\|_S \rightarrow 0 \text{ as } N \rightarrow \infty \text{ since } (\alpha_n) \in S. \text{ Then}$$

$$\left| \sum_{p=0}^{\infty} \alpha_p z^p - \sum_{p=0}^N \alpha_p \right| \leq \left| \sum_{p=0}^{N-1} \alpha_p z^p - \sum_{p=0}^{N-1} \alpha_p \right| + \left| \sum_{p=N}^{\infty} \alpha_p z^p - \sum_{p=N}^{\infty} \alpha_p \right| = A+B. \text{ For } \epsilon > 0,$$

fix  $N$  so that  $B \leq (2+C)\|(\alpha_p)_{p=N}^{\infty}\|_S \leq \epsilon/2$ . For  $|z-1|$  sufficiently small,

$A < \epsilon/2$ . Hence  $\lim_{z \rightarrow 1} \sum_{p=0}^{\infty} \alpha_p z^p = \sum_{p=0}^{\infty} \alpha_p$ . Therefore the theorem is proved.

## 4. THE DUAL OF J

Of course, for fixed  $s = (s_n)$  in  $S$  the formula  $\phi_s(\beta) = \sum_{j=0}^{\infty} s_j \beta_j$  defines a bounded linear functional on  $J$ . However, these are not all the members of  $J^*$ . As a key to our description of  $J^*$ , we note that if

$\beta = (\beta_n)$  is in  $J$  with limit  $\rho$  then  $\phi_s(\beta) = (\sum_{j=0}^{\infty} s_j) \rho + \sum_{j=0}^{\infty} s_j (\beta_j - \rho) = \lambda_0 \rho + \sum_{j=1}^{\infty} \lambda_j (\rho - \beta_{j-1})$ , where  $\lambda_0 = \sum_{j=0}^{\infty} s_j$  and for  $j \geq 1$   $\lambda_j = -s_{j-1}$ . Here  $\sum_{j=0}^{\infty} \lambda_j = 0$ ; so that  $\lambda$  is in  $S$ .

**THEOREM 4.1** (Holder's type inequality). If  $\lambda = (\lambda_n) \in H$ ,  $\beta = (\beta_n) \in J$ , and

$$\rho = \lim \beta \quad \text{then} \quad \left| \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) \right| \leq \|\lambda\|_H \cdot \|\beta\|_J.$$

*Proof:* This inequality follows easily from the observation that

$$\lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) = \lambda_0 \beta_0 + \sum_{n=1}^{\infty} \left( \sum_{j=0}^n \lambda_j \right) (\beta_n - \beta_{n-1});$$
 so that

$$\left| \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) \right| \leq \sup_{n \geq 0} \left| \sum_{j=0}^n \lambda_j \right| \left( |\beta_0| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| \right).$$

As we did for  $S$ , we consider the mapping  $\phi_\lambda: J \rightarrow \mathbb{R}$  defined

$$\text{by } \phi_\lambda(\beta) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1}) \quad \text{where } \lambda = (\lambda_n) \text{ is a fixed element in } H.$$

Theorem 4.1 assures us that  $\phi_\lambda$  is a bounded linear functional on  $J$ .

The claim of the next theorem is that these are all the bounded linear functionals on  $J$ . Here, our representation is in a slightly different sense than that of the Definition 2.1, the latter appearing more natural to us.

**THEOREM (4.2)** (Duality Theorem). If  $\phi \in J^*$  then there is a unique

$\lambda = (\lambda_n) \in H$  such that  $\phi = \phi_\lambda$ , that is,  $\phi(\beta) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n (\rho - \beta_{n-1})$  for any  $\beta = (\beta_n) \in J$ . Moreover  $\|\phi\|_{J^*} = \|\lambda\|_H$ . Conversely if  $\phi = \phi_\lambda$  then  $\phi \in J^*$ . So that the mapping  $\psi: H \rightarrow J^*$  defined by  $\psi(\lambda) = \phi_\lambda$  is an isometric isomorphism. Here,  $\rho = \lim \beta$ .

*Proof:* By the observation preceding the theorem we see that it remains to prove the first part. Let  $\phi \in J^*$ ; then in fact using Theorem 2.7

$$\phi(\beta) = \phi(L^{-1} \circ L(\beta)) = (\phi \circ L^{-1})(L(\beta)) = (\phi \circ L^{-1})((\beta_n - \beta_{n-1})_{n=0}^{\infty}), \quad \beta_{-1} = 0.$$

Now notice that  $\phi \circ L^{-1} \in \mathcal{L}_1^*$ , so that there is a  $\mu = (\mu_n) \in \mathcal{L}_\infty$  such that

$$(4.3) \quad \phi(\beta) = (\phi \circ L^{-1})((\beta_n - \beta_{n-1})_{n=0}^{\infty}) = \sum_{n=0}^{\infty} \mu_n (\beta_n - \beta_{n-1}).$$

By Theorem 2.8  $\mu_n = \sum_{j=0}^n \lambda_j$  for  $n \geq 0$  and some  $\lambda = (\lambda_j) \in H$ ;

consequently (4.3) becomes  $\phi(\beta) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^n \lambda_j \right) (\beta_n - \beta_{n-1})$ .

Observe now that

$$\sum_{n=0}^N \left( \sum_{j=0}^n \lambda_j \right) (\beta_n - \beta_{n-1}) = \lambda_0 \beta_0 + \sum_{n=1}^N \left( \sum_{j=0}^n \lambda_j \right) (\beta_n - \beta_{n-1}) =$$

$$\begin{aligned}
 &= \lambda_0 \beta_N + \sum_{n=1}^N \lambda_n [\beta_n - \beta_{n-1}] = \lambda_0 \beta_N + \sum_{n=1}^N \lambda_n [\beta_n - \rho + \rho - \beta_{n-1}] = \\
 &= \lambda_0 \rho + [\beta_N - \rho] \sum_{n=0}^N \lambda_n + \sum_{n=1}^N \lambda_n [\rho - \beta_{n-1}]. \text{ Taking the limit as } N \rightarrow \infty
 \end{aligned}$$

we get

$$\sum_{n=0}^{\infty} \left( \sum_{j=0}^{\infty} \lambda_j \right) (\beta_n - \beta_{n-1}) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n [\rho - \beta_{n-1}]$$

since  $[\beta_N - \rho] \sum_{n=0}^N \lambda_n \rightarrow 0$  as  $N \rightarrow \infty$  ( $\rho = \lim \beta$ ). Therefore we have

$$\phi(\beta) = \lambda_0 \rho + \sum_{n=1}^{\infty} \lambda_n [\rho - \beta_{n-1}].$$

Again by Theorem 2.8  $\|\phi \circ L^{-1}\|_{\ell_1^*} = \|\mu\|_{\ell_\infty} = \sup_{n \geq 0} |\mu_n| = \sup_{j=0}^n \left| \sum_{j=0}^n \lambda_j \right| = \|\lambda\|_H$  or  $\|\lambda\|_H = \|\phi \circ L^{-1}\|_{\ell_1^*} \leq \|\phi\|_{J^*} \cdot \|L^{-1}\| \leq \|\phi\|_{J^*}$  since  $\|L^{-1}\| \leq 1$ . On the other hand Theorem 4.1 gives us that  $|\phi(\beta)| \leq \|\lambda\|_H \|\beta\|_J$  (since  $\phi = \phi_\lambda$ ), which implies  $\|\phi\|_{J^*} \leq \|\lambda\|_H$ . Putting together these two inequalities we have  $\|\phi\|_{J^*} = \|\lambda\|_H$ , so that the theorem is proved.

5. CHARACTERIZATION OF OPERATORS ON S

In this section we characterize all bounded linear operators on S, even though an alternate proof can be given using the uniform boundedness principle. We rather use the characterization of operators on  $c_0$ . In fact we have the following result.

THEOREM 5.1. The linear mapping  $A:S \rightarrow c_0$  is bounded if and only if there

is an infinite matrix  $(a_{nk})$  such that for any  $s = (s_n) \in S$

$$As = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n} & \dots \\ a_{10} & a_{11} & a_{12} & \dots & a_{1n} & \dots \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} & \dots \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \cdot \\ s_n \\ \cdot \\ \cdot \end{bmatrix}$$

satisfying

- i)  $\lim_{n \rightarrow \infty} a_{nk} = 0$ , for any fixed  $k \geq 0$  i.e. the columns of the matrix are in  $c_0$ ,
- ii)  $\sup_{n \geq 0} \sum_{k=0}^{\infty} |a_{n(k-1)} - a_{nk}| < \infty$  i.e. the rows of the matrix are uniformly bounded in J.

Here  $a_{n(-1)} = 0$  for  $n = 0, 1, 2, 3, \dots$ .

Proof: Note that the operator  $P = AT^{-1}$  where  $T^{-1}$  is as in Theorem 2.3 maps  $c_0$  into  $c_0$ , that is  $P:c_0 \rightarrow S \rightarrow c_0$ , so that applying Theorem 2.5 we get the desired result.

THEOREM 5.2. The linear mapping  $B:S \rightarrow S$  is bounded if and only if B is represented by an infinite matrix such that for some  $A:S \rightarrow c_0$  we have

$B = A - \tilde{A}$ , where  $\tilde{A}$  is defined by  $\tilde{A} = (\tilde{a}_{nk})$  with  $\tilde{a}_{nk} = a_{(n+1)k}$ .

( $\tilde{A}$  is a shifting of  $A$  up by one row) Note that  $a_{nk} = \sum_{j=n}^{\infty} b_{jk}$ .

Proof: We define  $A: S \xrightarrow{B} S \xrightarrow{T} c_0$ , where  $T$  is as in Theorem 2.3. That is,  $A = TB$  so that  $B = T^{-1}A$ .

Now using the matrix representation for  $T^{-1}$  we get

$$B = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad A = [I - \tilde{I}]A = IA - \tilde{I}A = A - \tilde{A} \quad \text{where}$$

$I$  is the identity matrix and  $\tilde{I}$  is formed by the shifting of  $I$  up by one row.

Notice Theorem 5.2 is equivalent to saying that  $B: S \rightarrow S$  is a bounded operator if  $B$  is an infinite matrix such that each column of  $B$  is summable, and if we consider the matrix  $r(B)$  with  $r(B) = (y_{nk})$ , where  $y_{nk} = \sum_{j=n}^{\infty} b_{jk}$ , then then the rows of  $r(B)$  are uniformly bounded in  $J$ .

THEOREM 5.3. The linear mapping  $C: c_0 \rightarrow S$  is bounded if and only if  $C$  is represented by an infinite matrix such that for some  $M: c_0 \rightarrow c_0$  we have  $C = M - \tilde{M}$ ,  $\tilde{M}$  is a shifting of  $M$  up by one row.

Proof: Notice  $c_0 \xrightarrow{C} S \xrightarrow{T} c_0$ ,  $T$  as in Theorem 2.3. So define  $M = TC$ ; therefore  $C = T^{-1}M = M - \tilde{M}$ .

Note here that the  $M$ 's are special  $A$ 's as are the  $B$ 's. The  $C$ 's are special  $B$ 's. The reader might wish to compare these last three theorems with Exercises 45 and 46, pg. 77, of [1].

We would like to point out that the spaces  $H$  and  $J$  may also be found in [1] (pg. 240). Another space there, denoted by  $cs$ , is the space of all

summable sequences  $s = (s_j)$  normed with norm on  $H$ ,  $\max_{n \geq 0} \left| \sum_{j=0}^n s_j \right|$ . It may be shown, as pointed out there, that  $cs^*$  looks like  $J$ . We hasten here to note that  $J$  is not the dual of  $cs$  in the sense of Definition 2.1. In particular, the inequality  $\left| \sum_{j=0}^{\infty} s_j \beta_j \right| < \|s\|_{cs} \cdot \|\beta\|_J$  does not hold. Consider the example:

Let  $s = (s_n)$  be given by

$s_0 = 1, s_1 = s_2 = s_3 = s_4 = 0, s_5 = -2, s_6 = s_7 = \dots = 0$ ; also,  $\beta = (\beta_n)$ :

$\beta_0 = 0, \beta_1 = -1, \beta_2 = -2, \beta_3 = -3, \beta_4 = -4, \beta_5 = \beta_6 = \dots = -4$ .

Notice that  $\sum_{j=0}^{\infty} s_j \beta_j = 8, \|\beta\|_J = 4$  and  $\|s\|_{cs} = \max_{n \geq 0} \left| \sum_{j=0}^n s_j \right| = 1$ . Therefore

$\left| \sum_{j=0}^{\infty} s_j \beta_j \right| \leq \left[ \max_{n \geq 0} \left| \sum_{j=0}^n s_j \right| \right] \cdot \|\beta\|_J$  does not hold.

To illustrate further the difference between  $cs$  and  $S$ , we note that  $cs$  and  $S$  are equivalent Banach spaces: if  $s$  is in  $S$  then  $\|s\|_{cs} \leq 2\|s\|_S$

and  $\|s\|_S \leq 2\|s\|_{cs}$ . So that  $\phi$  is in  $S^*$  if and only if  $\phi$  is in  $cs^*$ .

For  $j \geq 0$  let  $e_j$  be the sequence  $\tau$  such that  $\tau_j = 1$  and if  $k \neq j$   $\tau_k = 0$ .

Now for  $\phi$  in  $S^*$ ,  $\phi(s) = \sum_{j=0}^{\infty} s_j \phi(e_j)$ ; the sequence  $\{\phi(e_j)\}_{j=0}^{\infty}$  is in  $J$  and  $\|\{\phi(e_j)\}\|_J = \|\phi\|_{S^*}$ . But if  $\beta$  is the sequence of the example and  $\phi$  is defined on  $S$  by  $\phi(s) = \sum_{j=0}^{\infty} \beta_j s_j$  (so that  $\beta_j = \phi(e_j)$ ) then  $\|\beta\|_J = 4$ . So the natural association of  $\phi$  in  $cs^*$  with the sequence  $\{\phi(e_j)\}$  in  $J$  is not an isometry. There is an isometry from  $cs^*$  onto  $J$ , obtained by noting that  $cs$  is isometrically isomorphic to the space  $c$  of all convergent sequences normed with norm in  $\ell_{\infty}$ ; the dual  $c^*$  being isometrically isomorphic to  $\ell_1$ . This representation of  $c^*$  as  $\ell_1$ , and also the representation of  $cs^*$  as  $J$ , is complicated by the fact that the sequence  $\{e_j\}$  does not have dense span in  $c$ . However, the latter representation (of  $cs^*$  as  $J$ ) is made somewhat simpler by applying Theorems 2.7 and 3.5. For  $s$  in  $S$  let  $S_N(s) = \sum_{j=0}^N s_j$ , the  $N$ th partial sum of  $s$ , and  $\sigma(s) = \sum_{j=0}^{\infty} s_j$ . Let  $\phi$  be in  $cs^*$  with  $\phi(e_j) = \beta_j$ . Now, with  $L$  as in Theorem 2.7, we have  $\phi(s) = \sum_{j=0}^{\infty} s_j \phi(e_j) = \sum_{j=0}^{\infty} \beta_j s_j = \beta_0 r_0(s) + \sum_{n=1}^{\infty} r_n(s) [\beta_n - \beta_{n-1}] = \beta_0 \sigma(s) + \sum_{n=1}^{\infty} [\sigma(s) - S_{n-1}(s)] [\beta_n - \beta_{n-1}] = \sigma(s) \nu_0 + \sum_{n=1}^{\infty} S_{n-1}(s) \nu_n$  ( $\nu_0 = \lim \beta$ ,  $\nu_n = \beta_{n-1} - \beta_n$ ); so  $\nu = L(\nu)$ , with  $\mu_0 = \lim \beta$ ,  $\mu_n = \lim \beta + \beta_0 - \beta_n$ . These computations suggest that if one uses (as in [1]) the space  $cs$  rather than  $S$  the norm to be used in  $J$  should be given by  $\|\beta\| = \left| \lim \beta \right| + \sum_{n=1}^{\infty} \left| \beta_n - \beta_{n-1} \right|$ . However, the norm used there is the same as ours (see 1, pg. 239). With this norm,  $\|\beta\|$ , on  $J$ , the Holder's type inequality we get is  $\left| \sum_{j=0}^{\infty} \alpha_j \beta_j \right| \leq \|\alpha\|_{cs} \|\beta\|$ ,  $\alpha$  in  $S$  and  $\beta$  in  $J$ .

The reader might wish to compare the remarks of this last paragraph with some of D. J. H. Garling in pages 999-1000 of [2] and page 964 of [3].

These ideas were suggested by a norm on certain power series spaces introduced by the latter author in an investigation [4] of Fourier sine series and Lidstone series.

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