

CERTAIN NEAR-RINGS ARE RINGS, II

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ABSTRACT. We investigate distributively-generated near-rings R which satisfy one of the following conditions: (i) for each $x, y \in R$, there exist positive integers m, n for which $xy = y^m x^n$; (ii) for each $x, y \in R$, there exists a positive integer n such that $xy = (yx)^n$. Under appropriate additional hypotheses, we prove that R must be a commutative ring.

KEY WORDS AND PHRASES. Commutativity, Distributively-generated near-rings.

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1. INTRODUCTION AND TERMINOLOGY.

Consider the following two properties, either of which is known to imply commutativity for rings R [3, 4]:

(C₁) For each $x, y \in R$, there exist positive integers $n = n(x, y)$ and $m = m(x, y)$ for which $xy = y^m x^n$.

(C₂) For each $x, y \in R$, there exists a positive integer $n = n(x, y)$ such that $xy = (yx)^n$.

The main purpose of this note is to show that certain distributively-generated (d-g) near-rings R with these properties must be rings. The work may be regarded as a continuation of that in [2], to which paper the reader is referred for basic definitions.

Throughout the paper, the symbols Z and Z_+ will denote the integers and the positive integers respectively. The additive group of the ring R will be denoted by R^+ , its derived subgroup by R' , and its center by $\xi(R)$. The two-sided annihilator of the subset S of R will be denoted by $A(S)$, and the ideal generated by multiplicative commutators by $C(R)$. Such terms as center, central, commute, and commutator, unless specifically stated to refer to addition, may be assumed to refer to multiplication. The near-ring R will be called strongly-distributively-generated (s-d-g) if it contains a set of distributive elements whose squares generate R^+ .

The principal results (Theorems 2 and 4) are the following:

- (a) Any s-d-g near-ring with 1 which satisfies (C₁) is a commutative ring.
- (b) Any d-g near-ring R which satisfies (C₂) and has $R^2 = R$ is a commutative ring.

For the proofs, we shall require the classical theorem of Fröhlich [5, p. 93] which asserts that a d-g near-ring is distributive if and only if R^2 is additively commutative. Moreover, we shall assume the easy result that in any d-g near-ring R , the derived subgroup R' of R^+ is an ideal.

2. A COMMUTATIVITY RESULT FOR ARBITRARY d-g (C_1) -NEAR-RINGS.

THEOREM 1. Let R be any d-g near-ring satisfying (C_1) . Then both $C(R)$ and R' are nil ideals. In particular, if R has no non-trivial nil ideals, then R is a commutative ring.

The proof follows from two lemmas.

LEMMA 1. If R is any d-g near-ring satisfying (C_1) , then R has each of the following properties:

- (a) If $a, b \in R$ and $ab = 0$, then $ba = 0 = axb$ for all $x \in R$.
- (b) All one-sided annihilators are two-sided and are ideals of R .
- (c) Idempotent elements of R are central.
- (d) The set N of nilpotent elements of R forms an ideal.

PROOF. Property (a) follows at once from (C_1) , and (b) follows easily from (a). To establish (c), let e be an idempotent and $x \in R$. Then for some $p, q \geq 1$, $xe = e^p x^q = ex^q$ and hence $exe = ex^q = xe$; similarly, $ex = x^r e$ for some $r \geq 1$, so that $exe = x^r e = ex$. Thus $ex = xe$.

To establish (d), it will suffice to show that N forms an additive subgroup [2, Lemma 1]; and this may be done by proving that for each $a, b \in N$ and each positive integer j , $(a-b)^j$ is a finite sum $\sum \pm p_i$, where each p_i is a finite product of elements of R , of which at least j belong to $T = \{a, b\}$. To see that this is enough, note that if $a^n = b^m = 0$ and if $j = n + m - 1$, then each of the summands p_i will be zero by (a).

We proceed by induction on j , the case $j = 1$ being trivial. Suppose the result holds for j and write

$$(a-b)^{j+1} = (a-b)(a-b)^j = (a-b)(\sum \pm p_i) = \sum \pm (a-b)p_i,$$

where each p_i is a product having at least j factors from T ; and for each $p = p_i$ write $(a-b)p = p^k(a-b)^m = (p^k a - p^k b)(a-b)^{m-1}$. If $m = 1$, we are finished; otherwise, express $(a-b)^{m-1}$ in the form $\sum \pm d_s$ where the d_s are distributive. Since $(a-b)p = \sum \pm (p^k a d_s - p^k b d_s)$ and since each of the products $p^k a d_s$ and $p^k b d_s$ has at least $j + 1$ factors from T , the inductive step, and hence the proof of (d), is complete.

In view of (d), the proof of Theorem 1 will be complete once we establish the following lemma.

LEMMA 2. Let R be a d-g near-ring satisfying (C_1) and having no non-zero nilpotent elements. Then R is a commutative ring.

PROOF. By Lemma 3 of [1], R is a subdirect product of homomorphic images having no zero divisors; thus we assume that R has no zero divisors. Note that if R is multiplicatively commutative, hence distributive, then R^2 is additively commutative; therefore, for all $a, b \in R$

$$0 = a^2 + ab - a^2 - ab = a(a+b-a-b) = a + b - a - b,$$

so that R^+ is abelian. Observe also that if e is any non-zero idempotent, the fact that $e(x - ex) = (x - xe)e = 0$ for all $x \in R$ shows that e must be a multiplicative identity element.

Assume, then, that R^+ is not abelian; hence R is not commutative. Let a and b be elements of R which do not commute; and let m, n, s, t be positive integers, at least one of which is greater than 1, for which $ab = b^m a^n = a^{ns} b^{mt}$. If $ns = 1$, then $a(b-b^{mt}) = 0$, and hence b^{mt-1} is a non-zero idempotent. If $ns > 1$, then $ab^v = a^{ns-1} ab^{mt} = (ab^{mt})^v (a^{ns-1})^w$ for appropriate positive integers v, w ; and there exists an element c , which is either a or an element of the form ya , such that $ab = abc$. It follows that c is idempotent, and incidentally that c must have been of the form ya .

So far we have shown that any non-central element a of R either has a left inverse or has the property that for any b not commuting with a , $b^k = 1$ for some positive integer k . Suppose the latter holds, let c_1 be an element not commuting with a , and use (C_1) to obtain c for which $ac_1 = ca$. It is easily verified that ca does not commute with a , so $(ca)^k = 1$ for some integer k ; and in this case also, a has a left inverse.

Now suppose that z is any non-zero central element, and that a is non-central. Then $az = za$ is also non-central, so az has a left inverse, and therefore z has a left inverse. Thus, R is a division near-ring; and since division near-rings have commutative addition, we have contradicted our initial assumption concerning R . Hence, R^+ is abelian and R is a ring. Multiplicative commutativity follows from the result of [3].

3. COMMUTATIVITY OF s - d - g (C_1) -NEAR-RINGS WITH 1.

The major theorem of this section is the following:

THEOREM 2. If R is a strongly-distributively-generated near-ring with 1 which has property (C_1) , then R must be a commutative ring.

Of course, it suffices to prove the theorem under the additional assumption that R is subdirectly irreducible (s - i). The lemmas which follow all treat the subdirectly irreducible case.

LEMMA 3. Let R be a s - i d - g near-ring with 1, in which all idempotents are central. Then 1 is the only non-zero idempotent.

PROOF. If e is any non-zero idempotent, the centrality of e enables us to show that $1-e$ is idempotent as well. Clearly $Re \subseteq A(1-e)$; moreover, if $x \in A(1-e)$, the representation $x = x(1-e) + xe$ shows that $x = xe \in Re$. Thus, $Re = A(1-e)$, and similarly $R(1-e) = A(e)$. But it is easy to show that $A(x)$ is an ideal for any central x , so Re and $R(1-e)$ are ideals, which obviously have trivial intersection. The subdirect irreducibility of R therefore forces $R(1-e)$ to be trivial, so that $e = 1$.

LEMMA 4. Let R be a s - i d - g (C_1) -near-ring with 1. Then

- (a) if $x \in R$, either x commutes with -1 or there exists $k \in \mathbb{Z}_+$ such that $x^k = 1$;
- (b) for each $x \in R$, $x^2(-1) = (-1)x^2$.

PROOF. (a) Suppose $x(-1) \neq (-1)x$, and choose $k, j, m, n \in \mathbb{Z}_+$ such that

$$x(-1) = (-1)^k x^j = x^{jn} (-1)^{km} \tag{3.1}$$

Assume first that $jn > 1$. If km is odd, we have $x = x^{jn}$, which implies that x^{jn-1} is a non-zero idempotent, necessarily equal to 1. If km is even, (3.1) yields $-x = x^{jn}$, hence $x = -x^{jn} = x^{jn-1} x(-1) = x^{jn-1} x^{jn}$ and $x^{2(jn-1)} = 1$.

On the other hand, if $jn = 1$, then (3.1) yields $-x = x(-1) = x$. Choose $q, s \in \mathbb{Z}_+$ with $(-1)x = x^s (-1)^q$. Since $x(-1) = x$, this implies $(-1)x = x^s$ with $s > 1$; and we conclude that $x = (-1)x^s = ((-1)x)x^{s-1} = x^{2s-1}$, so that $x^{2(s-1)} = 1$.

(b) It follows from part (a) that zero divisors commute with -1 , hence we may assume x is not a zero divisor. Now if $x(-1) = x^s$ for some $s \in \mathbb{Z}_+$, x commutes with $x(-1)$ and consequently $x(-1) = (-1)x$; therefore, we may assume that $x(-1) = (-1)x^s$ for some $s \in \mathbb{Z}_+$. Thus, $(-1)x(-1) = x^s$; and commuting x with $(-1)x(-1)$ gives $x(-1)x(-1) = (-1)x(-1)x$ or $(-x)^2 = (-1)(-x)(x)$. In this equality we may replace x by $-x$, since x commutes with -1 if and only if $-x$ does; therefore $x^2 = (-1)x(-x)$ and $(-1)x^2 = x(-x) = -x^2 = x^2(-1)$.

LEMMA 5. Let R be a s -i d -g (C_1) -near-ring with 1 , and let D be the set of zero divisors. Then

- (a) D is an ideal and D^+ is abelian;
- (b) $R' = C(R) \subseteq A(D)$;
- (c) if $d \in D$ and x does not commute with d , then there exists $s \in \mathbb{Z}_+$ such that $xd = dx^s$.

PROOF. (a) Let S be the heart of R — that is, the intersection of all non-zero ideals; to show D is an ideal, we show that $D = A(S)$. Clearly $A(S) \subseteq D$; conversely, if $d \in D$, $A(d)$ is a non-trivial ideal, hence $S \subseteq A(d)$ and $d \in A(S)$. Therefore $D = A(S)$. Note that by Lemma 4(a), all elements of D commute with -1 ; thus if $d_1, d_2 \in D$, we have $-d_1 - d_2 = (-1)(d_1 + d_2) = (d_1 + d_2)(-1) = -d_2 - d_1$, so D^+ is abelian.

(b) If $x, y \in R$ and $d \in D$, then dx and dy are in D ; hence $dx + dy - dx - dy = 0 = d(x + y - x - y)$, and $R' \subseteq A(D)$. By Fröhlich's theorem and the commutativity of rings satisfying (C_1) , a d -g (C_1) -near-ring with 1 is additively commutative if and only if it is multiplicatively commutative; and applying this observation to $R/C(R)$ and R/R' gives $C(R) = R'$.

(c) By part (b) we have $d(dy - yd) = 0$ — that is, $d^2y = dyd$ — for all $y \in R$; moreover, since $yd = dy_1$ for some $y_1 \in R$, we get $yd^2 = dy_1d = d^2y_1 = dyd$ as well. Thus, for all $d \in D$, all $y \in R$ and all integers $i \geq 2$, we have $d^i y = yd^i$. Suppose that $d \in D$ and $x \in R$ are such that $xd = d^m x^s$ for some $m > 1$, and choose n, q such that $dx = x^n d^q$. Then $xd = d^m x^s = d^{m-1}(dx)x^{s-1} = d^{m-1}x^n d^q x^{s-1} = d^{m+q-1}x^{n+s-1}$, and $dx = x^n d^q = x^{n-1}(xd)d^{q-1} = x^{n-1}d^m x^s d^{q-1} = x^{n+s-1}d^{m+q-1}$. Since $m + q - 1 > 1$, we thus have $xd = dx$, so our proof is finished.

LEMMA 6. Let R be s -i and d -g with 1 , and suppose R satisfies (C_1) . Then $2R' = 0$.

PROOF. By Theorem 1, $R' \subseteq D$. By Lemma 4(a), all elements of R' commute with -1 ; in particular, for all $x, y \in R$, $x + y - x - y + y - x - y + x = x + y - x - x - y + x$ commutes with -1 . Taking for x and y a pair r, s of distributive elements and simplifying, we get the result that $2(r + s - r - s) = 0$, from which it follows that $2R' = 0$.

LEMMA 7. For R a s -i d -g (C_1) -near-ring with 1 , $A(2) \subseteq \xi(R)$. In particular, $R' \subseteq \xi(R)$ and therefore, $2R \subseteq \xi(R)$.

PROOF. If $2 \notin D$, R^+ is abelian by Lemma 6; hence assume $2 \in D$. Let $x \in A(2)$; and by Lemma 5(c), choose $k \in \mathbb{Z}_+$ such that $(1 + x)2 = 2(1 + x)^k = 2$. Thus, $1 + x + 1 + x = 1 + 1$, which yields $x + 1 = 1 + x$. If $b^k = 1$ for some $k \in \mathbb{Z}_+$, we now get $b + x - b - x = b(1 + b^{k-1}x - 1 - b^{k-1}x) = 0$. By Lemma 4(a), the only elements b yet to be considered commute with -1 ; and since x commutes additively with b if and only if it commutes additively with $b + x$, we may assume $b + x$ commutes with -1 also. Since $x \in D$, x commutes with -1 , and it follows at once that $b + x = x + b$. Thus, $A(2) \subseteq \xi(R)$.

That $R' \subseteq \xi(R)$ is now clear from Lemma 6. Noting that $2R = R2$, we complete the proof of the lemma by showing that $x + xy = y + x + x$ for all $x, y \in R$. But since $y - x = -x + y + c$ for some $c \in R'$ and since $R' \subseteq \xi(R)$, we have $x + x + y - x - x - y = x + y + c - x - y = x + y - x - y + c = y + c - y + c = c + c = 0$, the last equality following from Lemma 6.

PROOF OF THEOREM 2. We need only show that R^+ is abelian, since the theorem then follows from Fröhlich's theorem and the theorem of [3]. Let r, s be arbitrary distributive elements of R . By Lemma 4(b), $(r + s)^2$ commutes with -1 , which means that

$$-r^2 - sr - rs - s^2 = -s^2 - rs - sr - r^2. \tag{3.2}$$

If we write $rs = c + sr$, where $c = rs - sr \in \xi(R)$, and recall that $2sr \in \xi(R)$, we can write (3.2) as

$$-r^2 - s^2 - 2sr - c = -s^2 - r^2 - 2sr - c.$$

It follows at once that $r^2 + s^2 = s^2 + r^2$, and the fact that R is strongly-distributively-generated implies that R^+ is abelian.

I conjecture that Theorem 2 remains true if R is merely assumed to be d-g rather than s-d-g, but a proof eludes me. However, all the machinery is in place to establish two interesting cases of the conjecture.

THEOREM 3. Let R be a d-g near-ring with 1, and suppose R satisfies one of the following specialized versions of (C_1) :

- (C_3) For each $x, y \in R$, there exists an integer $n = n(x, y) \geq 1$ for which $xy = yx^n$.
- (C_4) For each $x, y \in R$, either $xy = yx$ or there exist $m, n \in \mathbb{Z}_+$ with $m \geq 2$, such that $xy = y^m x^n$.

Then R is a commutative ring.

PROOF. Again we may assume that R is subdirectly irreducible and (by Lemma 6) that $2 \in D$. Arguments similar to that of Lemma 5(c) show that zero divisors are central, and commuting 2 with $r + s$ for arbitrary distributive r, s now shows R^+ is abelian.

4. COMMUTATIVITY OF d-g (C_2) -NEAR-RINGS.

THEOREM 4. If R is any d-g near-ring satisfying (C_2) , then R is commutative. Moreover, if $R^2 = R$, then R is a ring.

PROOF. Note first that idempotents are central, for if e is idempotent and $x \in R$, and if we choose $n, m \in \mathbb{Z}_+$ such that $ex = (xe)^n$ and $xe = (ex)^m$, right-multiplying the first of these equalities by e and left-multiplying the second by e yields $ex = exe = xe$.

Now if R is any d-g (C_2) -near-ring and $a, b \in R$ are such that $ab \neq ba$, there exist $m, n > 1$ such that

$$ab = (ba)^n \text{ and } ba = (ab)^m; \tag{4.1}$$

it follows that

$$ab = (ab)^{nm} \text{ and } ba = (ba)^{nm}, \tag{4.2}$$

and hence that $(ab)^{nm-1}$ and $(ba)^{nm-1}$ are both idempotent. In fact, if $(ab)^t = e$ is idempotent, (4.1) shows that $(ba)^t = (ab)^{mt} = e$, hence $(ab)^{nm-1}$ and $(ba)^{nm-1}$ are equal

to the same idempotent, say e_1 .

We treat first the case of R with 1 , and as usual consider the subdirectly irreducible case. If we suppose R contains a pair of non-commuting elements a, b and choose n, m as above, then Lemma 3 guarantees that $(ab)^{nm-1} = (ba)^{nm-1} = 1$, so that a and b are both invertible. We choose $q \in \mathbb{Z}_+$ such that $(b^{-1}a)b = (b(b^{-1}a))^q$, which reduces at once to $ab = ba^q$. Thus, R is a commutative ring by Theorem 3.

We now drop the hypothesis that R has 1 . Again suppose $ab \neq ba$ and let n, m and e_1 be as above. The near-ring e_1R is a d -g (C_2) -near-ring having e_1 as multiplicative identity, hence is commutative. Therefore $e_1ae_1b - e_1be_1a = 0 = e_1(ab - ba)$; and since $e_1 = (ab)^{nm-1} = (ba)^{nm-1}$, an appeal to (4.2) yields the contradiction $ab = ba$. Hence R must be commutative; and if $R^2 = R$, Fröhlich's theorem shows that R is a ring.

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