

**THE PACKING AND COVERING OF THE COMPLETE GRAPH I: THE FORESTS OF ORDER FIVE**

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**ABSTRACT.** The maximum number of pairwise edge disjoint forests of order five in the complete graph  $K_n$ , and the minimum number of forests of order five whose union is  $K_n$ , are determined.

**KEY WORDS AND PHRASES.** *Packing, Covering, Decomposition.*

**1980 MATHEMATICS SUBJECT CLASSIFICATION CODE.** 05C05.

1. INTRODUCTION

Graphs in this paper are finite with no multiple edges or loops. Beineke [1] defined the general covering (respectively, packing) problem as follows:

For a given graph  $G$  find the minimum (maximum) number of edge disjoint subgraphs of  $G$  such that each subgraph has a specified property  $P$  and the union of the subgraphs is  $G$ .

Solutions of these problems are known only for a few properties  $P$ , when  $G$  is arbitrary. In most cases  $G$  is taken to be the complete graph  $K_n$  or the complete bipartite graph  $K_{m,n}$  (for particular references one may look at Roditty [2]).

**DEFINITION:** The complete graph  $K_n$  is said to have a G-decomposition if it is the union of edge disjoint subgraphs each isomorphic to  $G$ . We denote such a decomposition by  $G|K_n$ .

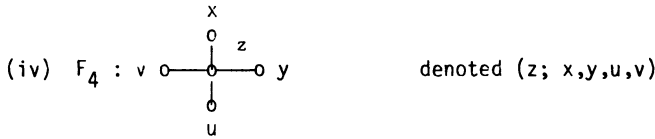
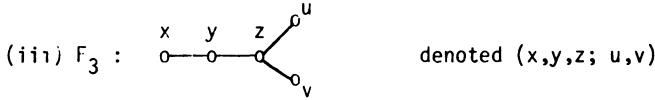
The G-decomposition problem is to determine the set  $N(G)$  of natural numbers such that  $K_n$  has a G-decomposition if and only if  $n \in N(G)$ . Note that G-decomposition is actually an exact packing and covering. In the proof of our problems of packing and covering, we make great use of the results obtained for the G-decomposition problem in cases when  $G$  has five vertices. As usual  $[x]$  will denote the largest integer not exceeding  $x$  and  $\{x\}$  the least integer not less than  $x$ . We will let  $e(G)$  denote the number of edges of the graph  $G$  and  $H = \bigcup_{i=1}^t G_i$

will show that the graph  $H$  is the union of  $t$  edge disjoint graphs  $G_i$ ,  $i=1,2,\dots,t$ .

The Theorem of this paper solves variations of the covering and packing problems for the four graphs below:

(i)  $F_1 : \begin{array}{cccc} x & y & z & u & v \\ \circ & \text{---} & \circ & \text{---} & \circ \\ \circ & & \circ & & \circ \end{array}$  denoted  $[(x,y,z)(u,v)]$

(ii)  $F_2 : \begin{array}{cccc} x & y & z & u & v \\ \circ & \text{---} & \circ & \text{---} & \circ \\ \circ & & \circ & & \circ \end{array}$  denoted  $[(x,y,z,u,v)]$



Our theorem may now be states as THEOREM (Packing and Covering).

let  $F$  be  $F_1, F_2$  or  $F_3$  and  $n \geq 5$  or  $F$  be  $F_4$  and  $n \geq 7$  then

(i) The maximum number of edge disjoint graphs  $F$  which are subgraphs of the complete graph  $K_n$  is

$$\lfloor e(K_n)/e(F) \rfloor.$$

(ii) The minimum number of graphs  $F$  whose union is the complete graph  $K_n$  is

$$\lceil e(K_n)/e(F) \rceil.$$

2. PROOF OF THE THEOREM

We give a separate proof for each choice of  $F$ .

$F_1$ : Proving the Theorem true for  $n \geq 5$  is a straightforward exercise. Bermond et al. [3] show that

$$N(F_1) = \{n | n \equiv 0, 1 \pmod{3}, n \geq 6\}. \tag{2.1}$$

Thus we have to consider only  $n = 3m + 2, m \geq 2$ . Observe that

$$K_{3m+2} = K_{3m} \cup K_{2,3m} \cup K_2, m \geq 2. \tag{2.2}$$

by (2.1)  $K_{3m}$  has an  $F_1$ -decomposition. Since  $K_{2,3m} = mK_{2,3}$  and  $K_{2,3}$  can be decomposed easily into two graphs  $F_1$ , it follows that  $K_{2,3m}$  has an  $F_1$ -decomposition. Only  $K_2$  in (2.2) is left non-packed. Hence, the Theorem is proved in this case.

$F_2$ : The proof will examine several cases depending on the value of  $n$ . The following table summarizes the cases  $n = 5, 6, 7, 8m$ , and  $8m + 1$  for  $m \geq 1$ .

n	packing	remains for covering
5	$(0,1,2,3,4); (1,3,0,4,2)$	$(0,2); (1,4)$
6	$(0,1,2,3,4); (0,5,4,1,3); (0,4,2,5,3)$	$(0,3), (0,2), (1,5)$
7	$(0,1,2,3,4); (0,2,4,6,1); (1,3,5,0,4)$ $(1,4,5,6,0); (1,5,2,6,3)$	$(0,3)$
$8m, 8m+1$	$F_2$ - decomposition [4]	

Table 1

We still have to prove the theorem for the cases:

$$n = 8m + k, \quad k = 2, \dots, 7$$

k = 2.

Let

$$K_{8m+2} = K_{8m} \cup K_{2,8m} \cup K_2. \quad (2.3)$$

The graph  $K_{8m}$  has an  $F_2$ -decomposition. since  $K_{2,8m} = 2mK_{2,4}$  and  $K_{2,4}$  can be decomposed easily into two graphs  $F_2$ , it follows that  $K_{2,8m}$  has an  $F_2$ -decomposition. Only  $K_2$  in (2.3) is left non-packed.

k = 3.

Let

$$K_{8m+3} = K_{8m+1} \cup K_{2,8m+1} \cup K_2. \quad (2.4)$$

The graph  $K_{8m+1}$  has an  $F_2$ -decomposition  $K_{2,8m+1} = K_{2,8m} \cup K_{2,1}$  and  $K_{2,8m}$  has an  $F_2$ -decomposition as we saw above. This decomposition of  $K_{2,8m}$  can be done in such a way that the edge  $(8m-1, 8m+2)$  is at one end of the  $F_2$  which includes it and the point  $8m-1$  is an end-point of that  $F_2$ . Thus we can replace the edge  $(8m-1, 8m+2)$  with the edge  $(8m, 8m+2)$ . Only the edges  $(8m, 8m+1)$ ,  $(8m+1, 8m+2)$ ,  $(8m-1, 8m+2)$  now remain non-packed, and they can be included in one more  $F_2$ .

k = 4.

Note that

$$K_{8m+4} = K_{8m} \cup K_{4,8m} \cup K_4. \quad (2.5)$$

The graph  $K_{8m}$  has an  $F_2$ -decomposition. Now

$$K_{4,8m} \cup K_4 = 2(2m-1)K_{2,4} \cup 2K_{2,4} \cup K_4 \quad (2.6)$$

and the  $2K_{2,4}$ 's can be selected to be vertex disjoint. Since  $K_{2,4}$  has an  $F_2$ -decomposition, so does  $2(2m-1)K_{2,4}$ . We need only to show that  $2K_{2,4} \cup K_4$  can be packed by 5  $F_2$  graphs, leaving two non-packed edges.

Let  $V(2K_{2,4}) = \{1, 2, \dots, 8, a, b, c, d\}$ ,  $V(K_4) = \{a, b, c, d\}$ .

Then, the 5 graphs of the packing of  $2K_{2,4} \cup K_4$  are:

$$(a, 3, b, c, 8); (1, a, d, 7, c); (4, b, d, c, 5); (a, c, 6, d, 8); (1, b, 2, a, 4).$$

The edges  $(d, 5)$  and  $(a, b)$  are left non-packed.

k = 5.

Let

$$K_{8m+5} = K_{8m+1} \cup K_{4,8m} \cup K_{4,1} \cup K_4. \quad (2.7)$$

The graph  $K_{8m+1}$  has an  $F_2$ -decomposition. In the case  $k = 4$  we saw that  $K_{4,8m} \cup K_4$  has an  $F_2$ -packing leaving two non-packed edges.

Let  $V(K_4) = \{a, b, c, d\}$  and  $V(K_{8m+1}) = Z_{8m+1}$ . Denote the non-packed edges by  $(a, b)$  and  $(8m-1, d)$ . We show that  $G = K_{4,1} \cup \{(a, b), (8m-1, d)\}$  has an  $F_2$ -packing leaving two non-packed edges. The  $F_2$  of this packing is  $(8m-1, d, 8m, a, b)$ . The non-packed edges are:  $(c, 8m)$  and  $(8m, b)$ .

k = 6.

Write

$$K_{8m+6} = K_{8m} \cup K_{6,8m} \cup K_6.$$

The graph  $K_{8m}$  has an  $F_2$ -decomposition. Observe that  $K_{6,8m} = 3K_{2,8m}$ . In the case  $k = 2$  we saw that  $F_2 | K_{2,8m}$ . Table 1 shows that  $K_6$  has  $F_2$ -packing leaving three non-packed edges as required, and these three can be included in one more  $F_2$ .

k = 7.

Let

$$K_{8m+7} = K_{8m+1} \cup K_{6,8m} \cup K_7.$$

The graph  $K_{8m+1}$  has an  $F_2$ -decomposition, and  $F_2 | K_{6,8m}$ , as was shown above. By Table 1 we know that the graph  $K_7$  has an  $F_2$ -packing leaving one non-packed edge. The Theorem has now been proved for  $F_2$  since all cases have been considered.

$F_3$ : The proof will consider the same cases as the proof for  $F_2$ .

n	packing	remains for covering
5	(0,1,2;3,4);(1,4,0;2,3)	(1,3),(3,4)
6	(0,1,2;3,4);(3,4,5;0,2);(0,3,1;4,5)	(0,2),(0,4),(3,5)
7	(3,2,0;1,6);(5,4,1;2,3);(1,6,3;4,5) (2,4,0;3,5);(1,5,6;2,4)	(2,5)
$8m, 8m+1$	$F_3$ - decomposition [4]	

Table 2

We now have to prove the theorem for the cases:

$$n = 8m+k, \quad k = 2, \dots, 7, \quad m \geq 1.$$

k = 2.

Let  $K_{8m+2}$  be as in (2.3). The graph  $K_{8m}$  has an  $F_3$ -decomposition. Since  $K_{2,8m} = 2mK_{2,4}$  and  $K_{2,4}$  can be decomposed easily into two  $F_3$  graphs, it follows that  $K_{2,8m}$  has an  $F_3$ -decomposition. Only  $K_2$  in (2.3) is left non-packed. Hence, the Theorem is proved in this case.

k = 3.

Let  $K_{8m+3}$  be as in (2.4).  $K_{8m+1}$  has an  $F_3$ -decomposition.  $K_{2,8m+1} = K_{2,8m} \cup K_{2,1}$ . The graph  $K_{2,8m}$  has an  $F_3$ -decomposition as was shown above. Replace the edge  $(8m-4, 8m+2)$  which appears in some  $F_3$  in the decomposition of  $K_{2,8m}$ , with the edge  $(8m, 8m+2)$ . Then the edges  $(8m-4, 8m+2)$ ,  $(8m+2, 8m+1)$ ,  $(8m+1, 8m)$  remain non-packed, but could be included in one additional  $F_3$ .

k = 4.

Let  $K_{8m+4}$  be as in (2.5). The graph  $K_{8m}$  has an  $F_3$ -decomposition. Let,  $K_{4,8m} \cup K_4$  be as in (2.6). Since  $K_{2,4}$  has an  $F_3$ -decomposition, so does  $2(2m-1)K_{2,4}$ . We show that  $2K_{2,4} \cup K_4$  can be packed by five  $F_3$  graphs, leaving two non-packed edges.

Let  $V(2K_{2,4}) = \{1,2,\dots,8,a,b,c,d\}$  and  $V(K_4) = \{a,b,c,d\}$ .  
Then the five graphs  $F_3$  are:

$$(c,6,d;7,8), (d,5,c;7,8), (4,a,c;d,b), (1,a,b;4,d), (a,3,b;1,2).$$

The edges  $(a,2)$  and  $(a,d)$  are left non-packed.

$k = 5$ .

Let  $K_{8m+5}$  be as in (2.7). The graph  $K_{8m+1}$  has an  $F_3$ -decomposition. In the case  $k = 5$  we saw that  $K_{4,8m} \cup K_4$  has an  $F_3$ -packing leaving two non-packed edges. Let  $V(K_4) = \{a,b,c,d\}$  and  $V(K_{8m+1}) = Z_{8m+1}$ . Denote the non-packed edges by  $(a,d)$  and  $(a,8m-1)$ . The  $F_3$  graph in  $K_{4,1} \cup \{(a,d), (a,8m-1)\}$  is  $(b,8m,a;d,8m-1)$ . The edges  $(d,8m)$  and  $(c,8m)$  remain non-packed.

The proofs for  $k = 6,7$  are accomplished in the same ways as for  $F_2$ .

Once again all cases have been considered and the proof is complete for  $F_3$ .

$F_4$ : it is easy to see that the theorem does not hold for  $n = 5$  and  $n = 6$ . For  $K_7$  the graphs  $F_4$  of the packing are:  $(0;1,2,3,4)$ ,  $(1;2,3,4,5)$ ,  $(2;3,4,5,6)$ ,  $(5;4,3,0,6)$ ,  $(6;0,1,3,4)$ . The edge  $(3,4)$  is left non-packed. Hence, the theorem is proved for  $n = 7$ . For  $n=8m, 8m+1$  we have an  $F_4$ -decomposition [5,6]. Hence, we again have to prove the theorem for the cases:

$$n = 8m + k, \quad k = 2, \dots, 7, \quad m \geq 1.$$

$k = 2$ .

Let  $K_{8m+2} = K_{8m+1} \cup K_{1,8m+1}$ . The graph  $K_{8m+1}$  has an  $F_4$ -decomposition.  $K_{1,8m+1}$  is a star that can easily be packed by  $2m$  stars  $F_4$ , leaving one non-packed edge.

$k = 3$ .

Let  $K_{8m+3} = K_{8m} \cup K_{3,8m} \cup K_3$ . The graph  $K_{8m}$  has an  $F_4$ -decomposition. Let  $K_{3,8m} = 3K_{1,8m}$ . Since the graph  $K_{1,8m}$  can be decomposed into  $2m$  stars  $F_4$ , it follows that  $K_{3,8m}$  also has an  $F_4$ -decomposition. Let  $V(K_3) = \{a,b,c\}$ , and create a decomposition of  $K_{3,8m}$  which includes the three stars  $(a;x,y,z,u)$ ,  $(b;x,y,z,u)$ ,  $(c;x,y,z,u)$ . Replace the edge  $(a,u)$  by  $(a,b)$ , the edge  $(b,u)$  by  $(b,c)$ , and the edge  $(c,u)$  by  $(c,a)$ . We did not spoil any star of the decomposition of  $K_{3,8m}$  and the star  $(u;a,b,c)$  of three branches is left non-packed.

$k = 4$ .

Let  $K_{8m+4} = K_{8m} \cup K_{4,8m} \cup K_4$ . The graph  $K_{8m}$  has an  $F_4$ -decomposition. Let  $K_{4,8m} = 4K_{1,8m}$ . The graph  $K_{1,8m}$  can be decomposed into  $2m$  stars  $F_4$  so  $K_{4,8m}$  has an  $F_4$ -decomposition. Let  $V(K_4) = \{a,b,c,d\}$ , and consider the subgraph  $K_{4,4}$  of  $K_{4,8m}$  whose vertices are given by  $V(K_{4,4}) = \{a,b,c,d\} \cup \{8m-1, 8m-2, 8m-3, 8m-4\}$ . The  $F_4$  decomposition of  $K_{4,8m}$  can be arranged in such a way that our  $K_{4,4}$  is made up of the four  $F_4$  graphs  $(a;8m-1,8m-2,8m-3,8m-4)$ ,  $(b;8m-1,8m-2,8m-3,8m-4)$ ,  $(c;8m-1,8m-2,8m-3,8m-4)$  and  $(d;8m-1,8m-2,8m-3,8m-4)$ . Replace the edges  $(a,8m-1)$ ,  $(b,8m-1)$ ,  $(c,8m-1)$ ,  $(d,8m-1)$  with the edges  $(a,c)$ ,  $(a,b)$ ,  $(b,c)$ ,  $(c,d)$ , respectively we now have a new  $F_4$  graph, namely  $(8m-1;a,b,c,d)$ . The edges  $(a,d)$  and  $(b,d)$  are the only one which remain non-packed.

$k = 5.$

Let  $K_{8m+5} = K_{8m} \cup K_{5,8m} \cup K_5$ . As before  $K_{8m}$  and  $K_{5,8m}$  have  $F_4$ -decompositions. Now  $K_5 = K_4 \cup K_{1,4}$  so we can complete the proof in the same way as in the case  $k = 4$ .

$k = 6.$

Let  $K_{8m+6} = K_{8m} \cup K_{6,8m} \cup K_6$ . The graphs  $K_{8m}$  and  $K_{6,8m}$  have  $F_4$ -decompositions. Let  $V(K_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ . Graph  $K_6$  can be packed with the two  $F_4$   $\{v_1; v_2, v_3, v_4, v_5\}$  and  $\{v_2; v_3, v_4, v_5, v_6\}$ . The induced graph on  $\{v_3, v_4, v_5, v_6\}$  is  $K_4$ . Hence, we can complete the proof here as in the case  $k = 4$ , leaving the edges  $(v_5, v_6)$ ,  $(v_4, v_6)$  non-packed. Those edges together with the non-packed edge  $(v_1, v_6)$  accomplish the proof of the theorem in this case.

$k = 7.$

Let  $K_{8m+7} = K_{8m} \cup K_{7,8m} \cup K_7$ . The graphs  $K_{8m}$  and  $K_{7,8m}$  have  $F_4$ -decompositions and we apply the  $F_4$ -packing shown for  $K_7$  at the beginning of this case.

This completes the proof of the theorem for  $F_4$ .  $\square$

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