

## CERTAIN SUBCLASSES OF BAZILEVIČ FUNCTIONS OF TYPE $\alpha$

**SHIGEYOSHI OWA**

Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577  
Japan

and

**MILUTIN OBRADOVIĆ**

Technološko-Matelurški Fakultet  
Kernegijeva 4  
11000 Beograd  
Yugoslavia

(Received October 25, 1985)

**ABSTRACT.** Certain subclasses  $B(\alpha, \beta)$  and  $B_1(\alpha, \beta)$  of Bazilevič functions of type  $\alpha$  are introduced. The object of the present paper is to derive a lot of interesting properties of the classes  $B(\alpha, \beta)$  and  $B_1(\alpha, \beta)$ .

**KEY WORDS AND PHRASES.** Bazilevič function, starlike function of order  $\beta$ , convex function of order  $\beta$ , subordination.

**1980 MATHEMATICS SUBJECT CLASSIFICATION.** 30C45.

### 1. INTRODUCTION.

Let  $A$  denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the unit disk  $U = \{z: |z| < 1\}$ . Let  $S$  be the subclass of  $A$  consisting of univalent functions in the unit disk  $U$ . A function  $f(z)$  belonging to the class  $A$  is said to be starlike of order  $\beta$  if and only if

$$\operatorname{Re}\{zf'(z)/f(z)\} > \beta \quad (1.2)$$

for some  $\beta$  ( $0 \leq \beta < 1$ ), and for all  $z \in U$ . We denote by  $S^*(\beta)$  the class of all functions in  $A$  which are starlike of order  $\beta$ . Throughout this paper, it should be understood that functions such as  $zf'(z)/f(z)$ , which have removable singularities at  $z = 0$ , have had these singularities removed in statements like (1.2). A function  $f(z)$  belonging to the class  $A$  is said to be convex of order  $\beta$  if and only if

$$\operatorname{Re}\{1 + zf''(z)/f'(z)\} > \beta \quad (1.3)$$

for some  $\beta$  ( $0 \leq \beta < 1$ ), and for all  $z \in U$ . Also we denote by  $K(\beta)$  the class of all functions in  $A$  which are convex of order  $\beta$ .

We note that  $f(z) \in K(\beta)$  if and only if  $zf'(z) \in S^*(\beta)$ . We also have  $S^*(\beta) \subseteq S^*(0) \equiv S^*$ ,  $K(\beta) \subseteq K(0) \equiv K$ , and  $K(\beta) \subset S^*(\beta)$  for  $0 \leq \beta < 1$ .

The classes  $S^*(\beta)$  and  $K(\beta)$  were first introduced by Robertson [1], and were studied subsequently by Schild [2], MacGregor [3], Pinchuk [4], Jack [5], and others.

A function  $f(z)$  of  $A$  is said to be in the class  $B(\alpha, \beta)$  if and only if

$$\operatorname{Re}\{zf'(z)f(z)^{\alpha-1}/g(z)^\alpha\} > \beta \quad (z \in U) \quad (1.4)$$

for some  $\alpha$  ( $\alpha > 0$ ) and for some  $\beta$  ( $0 \leq \beta < 1$ ), where  $g(z) \in S^*$ .

Furthermore, we denote by  $B_1(\alpha, \beta)$  the subclass of  $B(\alpha, \beta)$  for which  $g(z) \equiv z$ .

Note that  $B(0, 0) = B_1(0, 0) = S^*$ ,  $B(0, \beta) = B_1(0, \beta) = S^*(\beta)$ , and that  $B_1(1, \beta)$  is the subclass of  $A$  consisting of functions for which  $\operatorname{Re}\{f'(z)\} > \beta$  for  $z \in U$ .

The class  $B(\alpha, 0)$  when  $\beta = 0$  was studied by Singh [6] and Obradović ([7], [8]). Since  $B(\alpha, \beta) \subseteq B(\alpha, 0)$  for  $0 \leq \beta < 1$ , the class  $B(\alpha, \beta)$  is the subclass of Bazilevic functions of type  $\alpha$  (cf. [6]).

Let  $f(z)$  and  $g(z)$  be analytic in the unit disk  $U$ . Then a function  $f(z)$  is said to be subordinate to  $g(z)$  if there exists a function  $w(z)$  analytic in the unit disk  $U$  satisfying  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in U$ ) such that  $f(z) = g(w(z))$ . We denote by  $f(z) \prec g(z)$  this relation. In particular, if  $g(z)$  is univalent in the unit disk  $U$  the subordination is equivalent to  $f(0) = g(0)$  and  $f(U) \subset g(U)$ .

The concept of subordination can be traced back to Lindelöf [9], but Littlewood [10] and Rogosinski [11] introduced the term and discovered the basic relations.

2. SOME PROPERTIES OF THE CLASS  $B(\alpha, \beta)$ .

We begin to state the following lemma due to Miller and Mocanu [12].

LEMMA 1. Let  $M(z)$  and  $N(z)$  be regular in the unit disk  $U$  with  $M(0) = N(0) = 0$ , and let  $\beta$  be real. If  $N(z)$  maps  $U$  onto a (possibly many-sheeted) region which is starlike with respect to the origin then

$$\operatorname{Re}\{M'(z)/N'(z)\} > \beta \quad (z \in U) \implies \operatorname{Re}\{M(z)/N(z)\} > \beta \quad (z \in U), \tag{2.1}$$

and

$$\operatorname{Re}\{M'(z)/N'(z)\} < \beta \quad (z \in U) \implies \operatorname{Re}\{M(z)/N(z)\} < \beta \quad (z \in U). \tag{2.2}$$

Applying Lemma 1, we prove

LEMMA 2. Let the function  $f(z)$  defined by (1.1) be in the class  $S^*(\beta)$ , and let  $\alpha$  and  $c$  be positive integers. Then the function  $F(z)$  defined by

$$F(z)^\alpha = \frac{\alpha + c}{z^c} \int_0^z t^{c-1} f(t)^\alpha dt \quad (z \in U)$$

is also in the class  $S^*(\beta)$ .

PROOF. Setting

$$\frac{\alpha z F'(z)}{F(z)} = \frac{z^c f(z)^\alpha - c \int_0^z t^{c-1} f(t)^\alpha dt}{\int_0^z t^{c-1} f(t)^\alpha dt} = \frac{M(z)}{N(z)}, \tag{2.4}$$

we have  $M(0) = N(0) = 0$  and

$$\operatorname{Re}\{M'(z)/N'(z)\} = \alpha \operatorname{Re}\{z f'(z)/f(z)\} > \alpha \beta. \tag{2.5}$$

As  $N(z)$  is  $(\alpha+1)$ -valently starlike in the unit disk  $U$ , Lemma 1 shows that

$$\operatorname{Re}\{M(z)/N(z)\} = \alpha \operatorname{Re}\{zF'(z)/F(z)\} > \alpha\beta \quad (2.6)$$

which implies  $F(z) \in S^*(\beta)$ .

Now, we state and prove

**THEOREM 1.** Let the function  $f(z)$  defined by (1.1) be in the class  $B(\alpha, \beta)$  for  $g(z) \in S^*(\beta)$ , where  $\alpha$  is a positive integer and  $0 \leq \beta < 1$ . Then the function  $F(z)$  defined by (2.3) is also in the class  $B(\alpha, \beta)$ .

**PROOF.** It follows from (2.3) that

$$\frac{\alpha z F'(z)}{F(z)^{1-\alpha}} = \frac{\alpha + c}{z^c} \left\{ z^c f(z)^\alpha - c \int_0^z t^{c-1} f(t)^\alpha dt \right\}. \quad (2.7)$$

Note that there exists a function  $g(z)$  belonging to the class  $S^*(\beta)$  such that

$$\operatorname{Re}\{z f'(z) f(z)^{\alpha-1} / g(z)^\alpha\} > \beta. \quad (2.8)$$

Define the function  $G(z)$  by

$$G(z)^\alpha = \frac{\alpha + c}{z^c} \int_0^z t^{c-1} g(t)^\alpha dt. \quad (2.9)$$

Then, by using Lemma 2, we have  $G(z) \in S^*(\beta)$ . Combining (2.7) and (2.9), we observe that

$$\frac{\alpha z F'(z)}{F(z)^{1-\alpha} G(z)^\alpha} = \frac{z^c f(z)^\alpha - c \int_0^z t^{c-1} f(t)^\alpha dt}{\int_0^z t^{c-1} g(t)^\alpha dt}. \quad (2.10)$$

Setting

$$\frac{\alpha z F'(z)}{F(z)^{1-\alpha} G(z)^\alpha} = \frac{M(z)}{N(z)}, \quad (2.11)$$

(2.10) gives

$$\operatorname{Re}\{M'(z)/N'(z)\} = \alpha \operatorname{Re}\{zf'(z)f(z)^{\alpha-1}/g(z)^\alpha\} > \alpha\beta. \quad (2.12)$$

Consequently, with the help of Lemma 1, we conclude that

$$\operatorname{Re}\{zF'(z)F(z)^{\alpha-1}/G(z)^\alpha\} > \beta, \quad (2.13)$$

that is, that  $F(z) \in B(\alpha, \beta)$ . Thus we have Theorem 1.

COROLLARY 1. Let the function  $f(z)$  defined by (1.1) be in the class  $B(\alpha, 0)$ , where  $\alpha$  is a positive integer. Then the function  $F(z)$  defined by (2.3) is also in the class  $B(\alpha, 0)$ .

THEOREM 2. The set of all points  $\log\{z^{1-\alpha}f'(z)/f(z)^{1-\alpha}\}$ , for a fixed  $z \in U$  and  $f(z)$  ranging over the class  $B(\alpha, \beta)$ , is convex.

PROOF. We employ the same manner due to Singh [6]. For the function  $f(z)$  belonging to the class  $B(\alpha, \beta)$ , we define the function

$$h(z) = zf'(z)/f(z)^{1-\alpha}g(z)^\alpha, \quad (2.14)$$

where  $g(z) \in S^*$ . Then, it is clear that  $\operatorname{Re}\{h(z)\} > \beta$  for  $z \in U$ . We denote by  $P(\beta)$  the subclass of analytic functions  $h(z)$  satisfying  $\operatorname{Re}\{h(z)\} > \beta$  for  $0 \leq \beta < 1$  and  $z \in U$ . We note from (2.14) that

$$\log\{z^{1-\alpha}f'(z)/f(z)^{1-\alpha}\} = \log h(z) + \alpha \log\{g(z)/z\}. \quad (2.15)$$

Since, for a fixed  $z \in U$ , the range of  $\log h(z)$ , as  $h(z)$  ranges over the class  $P(\beta)$ , is a convex set, and the range of  $\log\{g(z)/z\}$ , as  $g(z)$  ranges over the class  $S^*$ , is a convex set, we complete the proof of Theorem 2.

Taking  $\alpha = 0$  in Theorem 2, we have

COROLLARY 2. The set of all points  $\log\{zf'(z)/f(z)\}$ , for a fixed  $z \in U$  and  $f(z)$  ranging over the class  $S^*(\beta)$ , is convex.

Furthermore, taking  $\alpha = 1$  in Theorem 2, we obtain

COROLLARY 3. The set of all points  $\log\{f'(z)\}$ , for a fixed  $z \in U$  and  $f(z)$  ranging over the class  $C(\beta)$ , is convex, where  $C(\beta)$  is the class of analytic functions  $f(z)$  which satisfy  $\operatorname{Re}\{zf'(z)/g(z)\} > \beta$  for  $g(z) \in S^*$

### 3. SOME PROPERTIES OF THE CLASS $B_1(\alpha, \beta)$ .

In order to derive some properties of the class  $B_1(\alpha, \beta)$ , we shall recall here the following lemmas.

LEMMA 3 (Miller [13]). Let  $\phi(u, v)$  be the complex function,  $\phi: D \rightarrow \mathbb{C}$ ,  $D \subset \mathbb{C} \times \mathbb{C}$  ( $\mathbb{C}$ -complex plane) and let  $u = u_1 + iu_2$ ,  $v = v_1 + iv_2$ . Suppose that the function  $\phi$  satisfies the conditions:

- (i)  $\phi(u, v)$  is continuous in  $D$ ;
- (ii)  $(1, 0) \in D$  and  $\operatorname{Re}\{\phi(1, 0)\} > 0$ ;
- (iii)  $\operatorname{Re}\{\phi(iu_2, v_1)\} \leq 0$  for all  $(iu_2, v_1) \in D$  and such that  $v_1 \leq -(1 + u_2^2)/2$ .

Let  $p(z) = 1 + p_1z + \dots$  be regular in the unit disk  $U$ , such that  $(p(z), zp'(z)) \in U$  for all  $z \in U$ . If  $\operatorname{Re}\{\phi(p(z), zp'(z))\} > 0$  ( $z \in U$ ), then  $\operatorname{Re}\{p(z)\} > 0$  for  $z \in U$ .

LEMMA 4 (Robertson [14]). Let  $f(z) \in S$ . For each  $0 \leq t \leq 1$  let  $F(z, t)$  be regular in the unit disk  $U$ , let  $F(z, 0) \equiv f(z)$  and  $F(0, t) \equiv 0$ . Let  $p$  be a positive real number for which

$$F(z) = \lim_{t \rightarrow +0} \frac{F(z, t) - F(z, 0)}{zt^p}$$

exists. Let  $F(z, t)$  be subordinate to  $f(z)$  in  $U$  for  $0 \leq t \leq 1$ , then

$$\operatorname{Re}\{F(z)/f'(z)\} \leq 0 \quad (z \in U). \quad (3.1)$$

If in addition  $F(z)$  is also regular in the unit disk  $U$  and  $\operatorname{Re}\{F(0)\} \neq 0$ , then

$$\operatorname{Re}\{F(z)/f'(z)\} < 0 \quad (z \in U). \quad (3.2)$$

LEMMA 5 (MacGregor [15]). Let the function  $f(z)$  be in the class  $K(\beta)$ . Then  $f(z) \in S^*(\gamma(\beta))$ , where

$$\gamma(\beta) = \begin{cases} \frac{2\beta - 1}{2(1 - 2^{1-2\beta})} & (\beta \neq 1/2) \\ \frac{1}{2 \log 2} & (\beta = 1/2). \end{cases} \quad (3.3)$$

We begin with

LEMMA 6. Let the function  $f(z)$  be in the class  $B_1(\alpha, \beta)$ , where  $\alpha$  is a positive integer and  $0 \leq \beta < 1$ . Then

$$\operatorname{Re}\{f(z)/z\}^\alpha > \beta \quad (z \in U). \quad (3.4)$$

PROOF. For  $f(z) \in B_1(\alpha, \beta)$ , we have

$$\operatorname{Re}\{zf'(z)f(z)^{\alpha-1}/z^\alpha\} = \operatorname{Re} \left\{ \frac{df(z)^\alpha/dz}{dz^\alpha/dz} \right\} > \beta. \quad (3.5)$$

Applying Lemma 1, we can prove the assertion (3.4).

THEOREM 3. Let the function  $f(z)$  be in the class  $B_1(\alpha, \beta)$ , where  $\alpha$  is a positive integer and  $0 \leq \beta < 1$ . Then the function  $F_1(z)$  defined by

$$F_1(z)^{\alpha+\gamma} = z^\gamma f(z)^\alpha \quad (3.6)$$

belongs to the class  $B_1(\alpha+\gamma, \beta)$  for  $\gamma \geq 0$ .

PROOF. Note that

$$\frac{(\alpha + \gamma)F_1'(z)}{F_1(z)^{1-(\alpha+\gamma)}} = \gamma z^{\gamma-1} f(z)^\alpha + \frac{\alpha z^\gamma f'(z)}{f(z)^{1-\alpha}}, \quad (3.7)$$

or

$$\frac{(\alpha + \gamma)zF_1'(z)}{F_1(z)^{1-(\alpha+\gamma)}z^{\alpha+\gamma}} = \gamma \left( \frac{f(z)}{z} \right)^\alpha + \frac{\alpha z f'(z)}{f(z)^{1-\alpha} z^\alpha}. \quad (3.8)$$

Therefore, by using Lemma 6, we have

$$\operatorname{Re}\{zF_1'(z)F_1(z)^{(\alpha+\gamma)-1}/z^{\alpha+\gamma}\} > \beta \quad (3.9)$$

which implies  $F_1(z) \in B_1(\alpha+\gamma, \beta)$ . Thus we complete the theorem.

Applying Lemma 3, we derive

**THEOREM 4.** Let the function  $f(z)$  be in the class  $B_1(\alpha, \beta)$  with  $\alpha > 0$  and  $0 \leq \beta < 1$ . Then

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\}^\alpha > \frac{1+2\alpha\beta}{1+2\alpha} \quad (z \in U). \quad (3.10)$$

**PROOF.** We define the function  $p(z)$  by

$$A\{f(z)/z\}^\alpha = p(z) + B, \quad (3.11)$$

where  $A = (1+2\alpha)/2\alpha(1-\beta)$  and  $B = (1+2\alpha\beta)/2\alpha(1-\beta)$ . Then  $p(z)$  is analytic in the unit disk  $U$  and  $p(0) = 1$ . Differentiating both sides of (3.11) logarithmically, we obtain

$$\text{or} \quad \frac{zf'(z)}{f(z)} = \frac{1}{\alpha} \left\{ \frac{zp'(z)}{p(z)+B} + \alpha \right\}, \quad (3.12)$$

$$zf'(z)f(z)^{\alpha-1}/z^\alpha = \{zp'(z) + \alpha(p(z)+B)\}/\alpha A. \quad (3.13)$$

Since  $f(z) \in B_1(\alpha, \beta)$ , (3.13) gives

$$\operatorname{Re}\{zp'(z) + \alpha(p(z)+B)\} - \alpha\beta A > 0. \quad (3.14)$$

Letting  $p(z) = u = u_1 + iu_2$  and  $zp'(z) = v = v_1 + iv_2$ , we consider the function

$$\phi(u, v) = v + \alpha(u + B) - \alpha\beta A \quad (3.15)$$

which is continuous in  $D = C \times C$ , and which  $(1, 0) \in D$  and

$\operatorname{Re}\{\phi(1, 0)\} = 3/2 > 0$ . Then, for all  $(iu_2, v_1)$  such that  $v_1 \leq -(1 + u_2^2)/2$ , we have

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= v_1 + \alpha\beta - \alpha\beta A \\ &\leq -u_2^2/2 \\ &\leq 0. \end{aligned} \tag{3.16}$$

Consequently, with the aid of Lemma 3, we conclude that

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in U), \tag{3.17}$$

that is, that

$$\operatorname{Re}\left\{ A \left[ \frac{f(z)}{z} \right]^\alpha \right\} > B. \tag{3.18}$$

This completes the proof of Theorem 4.

Putting  $\beta = 0$  in Theorem 4, we have

COROLLARY 3 ([8, Theorem 3]). Let the function  $f(z)$  be in the class  $B_1(\alpha, 0)$  with  $\alpha > 0$ . Then

$$\operatorname{Re}\left\{ \frac{f(z)}{z} \right\}^\alpha > \frac{1}{1 + 2\alpha} \quad (z \in U). \tag{3.19}$$

Taking  $\alpha = 1$  in Theorem 4, we have

COROLLARY 4. If the function  $f(z)$  belonging to  $A$  satisfies  $\operatorname{Re}\{f'(z)\} > \beta$  with  $0 \leq \beta < 1$ , then

$$\operatorname{Re}\left\{ \frac{f(z)}{z} \right\} > \frac{1 + 2\beta}{3} \quad (z \in U). \tag{3.20}$$

REMARK 1. Letting  $\beta = 0$  in Corollary 4, we have the corresponding result due to Obradović [7, Theorem 2].

Next, we prove

THEOREM 5. Let  $\alpha > 1$ ,  $0 \leq \beta < 1$ , and  $\gamma(\beta)$  define by (3.3). Let  $-1/4 \leq \alpha - \beta - (\alpha - 1)\gamma(\beta) \leq 1/4$ . If the function  $f(z)$  belongs to the class  $K(\beta)$ , then  $f(z) \in B_1(\alpha, \beta')$ , where

$$\beta' = 1/[2\{\alpha - \beta - (\alpha - 1)\gamma(\beta)\} + 1].$$

PROOF. Define the function  $p(z)$  by

$$Azf'(z)f(z)^{\alpha-1}/z^\alpha = p(z) + A - 1, \quad (3.21)$$

where  $A = 1 + 1/2\{\alpha - \beta - (\alpha - 1)\gamma(\beta)\}$ . Differentiating both sides of (3.21) logarithmically, we know that

$$1 - \alpha + \frac{zf''(z)}{f'(z)} + (\alpha - 1) \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z) + A - 1}, \quad (3.22)$$

or

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} - \beta + (\alpha - 1) \left[ \frac{zf'(z)}{f(z)} - \gamma(\beta) \right] \\ = \frac{zp'(z)}{p(z) + A - 1} + \alpha - \beta - (\alpha - 1)\gamma(\beta). \end{aligned} \quad (3.23)$$

With the help of Lemma 5, (3.23) implies

$$\operatorname{Re} \left\{ \frac{zp'(z)}{p(z) + A - 1} \right\} + \alpha - \beta - (\alpha - 1)\gamma(\beta) > 0. \quad (3.24)$$

Let the function  $\phi(u, v)$  be defined by

$$\phi(u, v) = \frac{v}{u + A - 1} + \alpha - \beta - (\alpha - 1)\gamma(\beta) \quad (3.25)$$

with  $p(z) = u = u_1 + iu_2$  and  $zp'(z) = v = v_1 + iv_2$ . Then  $\phi(u, v)$  is continuous in  $D = (C - \{1-A\}) \times C$ . Further,  $(1, 0) \in D$  and

$$\begin{aligned} \operatorname{Re}\{\phi(1, 0)\} &= \alpha - \beta - (\alpha - 1)\gamma(\beta) \\ &> (\alpha - 1)\{1 - \gamma(\beta)\} \\ &> 0. \end{aligned} \quad (3.26)$$

Consequently, for all  $(iu_2, v_1)$  such that  $v_1 \leq -(1 + u_2^2)/2$ , we obtain

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= \frac{(A - 1)v_1}{(A - 1)^2 + u_2^2} + \alpha - \beta - (\alpha - 1)\gamma(\beta) \\ &\leq \frac{2\{\alpha - \beta - (\alpha - 1)\gamma(\beta)\}\{(A - 1)^2 + u_2^2\} - (A - 1)(1 + u_2^2)}{2\{(A - 1)^2 + u_2^2\}} \\ &\leq 0. \end{aligned} \tag{3.27}$$

By virtue of Lemma 3, we have

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in U),$$

that is,

$$\operatorname{Re}\{Azf'(z)f(z)^{\alpha-1}/z^\alpha\} > A - 1. \tag{3.28}$$

It follows from (3.28) that

$$\operatorname{Re} \left\{ \frac{zf'(z)f(z)^{\alpha-1}}{z^\alpha} \right\} > \frac{1}{2\{\alpha - \beta - (\alpha - 1)\gamma(\beta)\} + 1} \tag{3.29}$$

which completes the assertion of Theorem 5.

Finally, we prove

**THEOREM 6.** Let  $f(z) \in A$ ,  $\alpha > 0$ ,  $0 \leq \beta < 1$ , and  $0 \leq t \leq 1$ . If

$$g(z) = \int_0^z \left[ \frac{f(s)}{s} \right]^{1-\alpha} ds \in S \tag{3.30}$$

and

$$\begin{aligned} G(z, t) &= f((1-t)z) - f(1-t^2)z \\ &+ (1 - t^2) \int_0^z \left[ \frac{f(s)}{s} \right]^{1-\alpha} ds + zt\beta \left[ \frac{f(z)}{z} \right]^{1-\alpha} \prec g(z), \end{aligned} \tag{3.31}$$

then  $f(z) \in B_1(\alpha, \beta)$ .

PROOF. Note that

$$\begin{aligned} G(z) &= \lim_{t \rightarrow +0} \frac{G(z, t) - G(z, 0)}{zt} \\ &= \lim_{t \rightarrow +0} \frac{\partial G(z, t) / \partial t}{z} \\ &= \beta \left\{ \frac{f(z)}{z} \right\}^{1-\alpha} - f'(z) \end{aligned} \quad (3.32)$$

and  $g'(z) = \{f(z)/z\}^{1-\alpha}$ . It is clear from (3.32) that  $\operatorname{Re}\{G(0)\} = \beta - 1 \neq 0$ . Consequently, applying Lemma 4 when  $p = 1$ , we have

$$\operatorname{Re} \left\{ \beta - \frac{zf'(z)f(z)^{\alpha-1}}{z^\alpha} \right\} < 0, \quad (3.33)$$

or

$$\operatorname{Re}\{zf'(z)f(z)^{\alpha-1}/z^\alpha\} > \beta \quad (3.34)$$

which shows  $f(z) \in B_1(\alpha, \beta)$ .

REMARK 2. Letting  $\beta = 0$  in Theorem 6, we have the corresponding theorem by Obradović [8, Theorem 1].

#### REFERENCES

1. Robertson, M. S., On the theory of univalent functions, Ann. of Math. 37(1936), 374 - 408.
2. Schild, A., On starlike functions of order  $\alpha$ , Amer. J. Math. 87 (1965), 65 - 70.
3. MacGregor, T. H., The radius of convexity for starlike functions of order  $1/2$ , Proc. Amer. Math. Soc. 14(1963), 71 - 76.

4. Pinchuk, B., On starlike and convex functions of order  $\alpha$ , Duke Math. J. 35(1968), 721 - 734.
5. Jack, I. S., Functions starlike and convex of order  $\alpha$ , J. London Math. Soc. (2)3(1971), 469 - 474.
6. Singh, R., On Bazilevič functions, Proc. Amer. Math. Soc. 38(1973), 261 - 271.
7. Obradović, M., Estimates of the real part of  $f(z)/z$  for some classes of univalent functions, Mat. Vesnik 36(1984), 266 - 270.
8. Obradović, M., Some results on Bazilevič functions, Mat. Vesnik 37(1985), 92 - 96.
9. Lindelöf, E., Mémoire sur certaines inégalités dans la théorie des fonctions monogènes et sur quelques propriétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel, Acta Soc. Sci. Fenn. 35(1909), 1 - 35.
10. Littlewood, J. E., On inequalities in the theory of functions, Proc. London Math. Soc. (2)23(1925), 481 - 519.
11. Rogosinski, W., On subordinate functions, Proc. Cambridge Philos. Soc. 35(1939), 1 - 26.
12. Miller, S. S. and Mocanu, P. T., Second order differential inequalities in the complex plane, J. Math. Ana. Appl. 65(1978), 289 - 305.
13. Miller, S. S., Differential inequalities and Caratheodory functions, Bull. Amer. Math. Soc. 81(1975), 79 - 81.
14. Robertson, M. S., Application of subordination principle to univalent functions, Pacific J. Math. 11(1961), 315 - 324.
15. MacGregor, T. H., A subordination for convex function of order  $\alpha$ , J. London Math. Soc. (2)9(1975), 530 - 536.

## Special Issue on Decision Support for Intermodal Transport

### Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/jamds/guidelines.html>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/>, according to the following timetable:

Manuscript Due	June 1, 2009
First Round of Reviews	September 1, 2009
Publication Date	December 1, 2009

### Lead Guest Editor

**Gerrit K. Janssens**, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; [Gerrit.Janssens@uhasselt.be](mailto:Gerrit.Janssens@uhasselt.be)

### Guest Editor

**Cathy Macharis**, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; [Cathy.Macharis@vub.ac.be](mailto:Cathy.Macharis@vub.ac.be)