

A NOTE ON THE INVERSE FUNCTION THEOREM OF NASH AND MOSER

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ABSTRACT. The Nash-Moser inverse function theorem is proved for different kind of differentiabilitys.

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1. INTRODUCTION

The purpose of this note is to formulate the inverse function theorem of Nash and Moser for different differentiabilitys using a categorical approach. The proof is based on the inverse function theorem of Nash and Moser in the version of Hamilton [1] formulated in the category of graded Fréchet spaces which admit smoothing operators and C_c^∞ -differentiable [2] tame maps. Our proof is using the same technique as Schmid [3] uses when he proves this theorem for a stronger notion of differentiability, called the Γ -differentiability, than the C_c^∞ -differentiability. From our formulation it is possible to derive the inverse function theorem of Nash and Moser for natural differentiabilitys stronger than the C_c^∞ -differentiability.

2. THE INVERSE FUNCTION THEOREM OF NASH AND MOSER.

Let LC denote the category of locally convex limit vector spaces [2] and continuous linear mappings. Further let K_α denote a coreflective subcategory of LC which is closed under finite products and the coreflector $?^\alpha: LC \rightarrow K_\alpha$ is the identity on morphisms and such that the identity mapping $(C_c(X,F))^\alpha = C_\alpha(X,F) \rightarrow C_c(X,F)$ is continuous. Here $C_c(X,F)$ denotes the vector space of continuous mappings $X \rightarrow F$, endowed with continuous convergence [2], and X is a limit space and $F \in \text{obj}(LC)$.

For any pair $E, F \in \text{obj}(LC)$ we let $L_c^k(E, F)$ be the space of all continuous k -linear mappings from E^k into F , endowed with continuous convergence. We write $(L_c^k(E, F))^\alpha = L_\alpha^k(E, F)$.

DEFINITION. Let E and F be locally convex spaces and let U be open in E .

A mapping $f : U \rightarrow F$ is said to be *differentiable of class C_α^p* , if there exist functions

$$D^k f : U \rightarrow L^k(E, F), \quad k = 0, 1, \dots, p,$$

such that $D^0 f = f$ and for each $x \in U$, each $h \in E$ and each $k = 0, 1, \dots, p-1$, we have

$$\lim_{t \rightarrow 0} t^{-1} (D^k f(x+th) - D^k f(x)) = D^{k+1} f(x)h,$$

and such that for each $k \in \mathbb{N}$, $k \leq p$, the following two conditions are satisfied:

- (1) $D^k f(U) \subseteq L^k(E, F)$
- (2) $D_\alpha^k f : U \rightarrow L^k(E, F)$ is continuous.

f is called *differentiable of class C_α^∞* if it is differentiable of class C_α^p for every $p \in \mathbb{N}$.

By Keller [2] the chain rule is valid for C_α^∞ , since α is a finer limit structure than continuous convergence. From the universal property of continuous convergence follows that for any continuous map $g : U \rightarrow L_\alpha^k(E, F)$ the associated map $\tilde{g} : U \times E^k \rightarrow F$ defined by $\tilde{g}(x, h_1, \dots, h_k) = g(x)(h_1, \dots, h_k)$, $x \in U$, $h_i \in E$, is continuous. As the limit structure α is always finer than c , we have that differentiability of class C_α^∞ implies differentiability of class C_c^∞ . The latter is exactly the concept of differentiability used by Hamilton [1] to prove the inverse function theorem of Nash and Moser.

We first recall some definitions that will be needed.

Let E be a Fréchet space. A grading on E is an increasing sequence of norms $(\|\cdot\|_r^1)_{r \in \mathbb{N}}$ on E which defines the topology on E . Two gradings $(\|\cdot\|_r^1)_{r \in \mathbb{N}}$ and $(\|\cdot\|_r^2)_{r \in \mathbb{N}}$ are equivalent if for some $s \in \mathbb{N}$ $\|x\|_r^1 \leq c\|x\|_{r+s}^2$ and $\|x\|_r^2 \leq c\|x\|_{r+s}^1$, $x \in E$, with a constant c which may depend on r . A graded space is a Fréchet space together with an equivalence class of gradings. We say that a graded space E admits smoothing operators if we can find linear maps $S_t : E \rightarrow E$, $1 \leq t < \infty$, such that for some r $\|S_t(x)\|_{i+k} \leq ct^{r+k}\|x\|_i$ and $\|S_t(x) - x\|_i \leq ct^{r-k}\|x\|_{i+k}$ for all $i, k \in \mathbb{N}$, $1 \leq t < \infty$, $x \in E$ and some constant c which may depend on i and k . Let E and F be graded spaces and U open in E . We say that a map $f : U \rightarrow F$ is tame if for every $x_0 \in U$ we can find a neighbourhood U_0 and a number $r \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have the growth estimate $\|f(x)\|_n \leq c(\|x\|_{n+r} + 1)$ for all $x_0 \in U$, where the constant c may depend on n .

In the proof of the inverse function theorem of Nash and Moser we shall also need the following result (Lemma 2, [3]): The composition of two continuous tame maps is continuous and tame.

THEOREM. Let E and F be graded spaces which admit smoothing operators. Let U be open in E and assume that

- (1) $f : U \rightarrow F$ is differentiable of class C_α^∞ and tame.
- (2) $\widetilde{D^k f} : U \times E^k \rightarrow F$ is tame for every $k \in \mathbb{N}$.
- (3) For each $x \in U$ the derivative $Df(x) : E \rightarrow F$ is an isomorphism.
- (4) The map $Vf : U \rightarrow L_\alpha(F, E)$, $Vf(x) = (Df(x))^{-1}$, is continuous.
- (5) $\widetilde{Vf} : U \times F \rightarrow E$ is tame.

Then for any $x_0 \in U$ we can find open neighbourhoods of x_0 and V_0 of $f(x_0)$ such that f is a bijective map from U_0 onto V_0 and the inverse map $f^{-1} : V_0 \rightarrow U_0$ is differentiable of class C_α^∞ and the maps $D^k f^{-1} : V_0 \times F^k \rightarrow E$ are tame for all $k \in \mathbb{N}$. Furthermore we have the formula $D(f^{-1})(y) = Vf(f^{-1}(y))$ for all $y \in V_0$.

PROOF. The maps $D^k f : U \times E^k \rightarrow F$ are continuous and tame, since f is differentiable of class C_α^∞ and assumption (2). Further the assumptions (4) and (5) imply that also $\widetilde{Vf} : U \times F \rightarrow E$ is continuous and tame. Now we have that f is differentiable of class C_α^∞ and all $D^k f$ are tame, $Df(x) : E \rightarrow F$ is an isomorphism for every $x \in U$ and the family of inverses $\widetilde{Vf} : U \times F \rightarrow E$ are continuous and tame maps. Consequently the conditions of the inverse function theorem of Nash-Moser are fulfilled (theorem 1.1.1 p. 171 in [1]). Then for every $x_0 \in U$ there exist neighbourhoods U_0 of x_0 and V_0 of $f(x_0)$ such that $f : U_0 \rightarrow V_0$ is bijective and $f^{-1} : V_0 \rightarrow U_0$ is continuous and tame. Furthermore the formula $\lim_{t \rightarrow 0} t^{-1}(f^{-1}(y + tw) - f^{-1}(y)) = Vf(f^{-1}(y))w$ holds, for all $y \in V_0$ and $w \in F$, by the proof of theorem 1.1.1 p. 186 in [1]. By induction on k we will prove the remaining part that $f^{-1} : V_0 \rightarrow U_0$ is differentiable of class C_α^∞ and $D^k f^{-1} : V_0 \times F^k \rightarrow E$ is tame for every $k \in \mathbb{N}$. From the formula $Df^{-1} = Vf \circ f^{-1}$ and assumption (4) follow that $Df^{-1} : V_0 \rightarrow L(F, E)$ is continuous. Further we have that $\widetilde{Df^{-1}} : V_0 \times F \rightarrow E$ is tame since \widetilde{Vf} and f^{-1} are tame. Assume now it to be true for k . From the definition of the α -differentiability follows that the map f^{-1} is C_α^{k+1} if Df^{-1} is differentiable of class C_α^k . Since $Df^{-1} = Vf \circ f^{-1}$, $D^{k+1} f^{-1}$ is clearly tame so we only have to show that Vf is differentiable of class C^k . By induction on p . By theorem 5.3.1, p. 102 in [1] we have that Vf is weakly differentiable and that $D(\widetilde{Vf}) : U_0 \times E \times F \rightarrow E$ is continuous and the formula $[D(\widetilde{Vf})](x)\{u, w\} = -Vf(x)[D^2 f(x)\{u, Vf(x)w\}]$ holds for all $x \in U_0$, $u \in E$ and $w \in F$. Thus the derivative $D(Vf) : U_0 \rightarrow L_\alpha(E \times F, E)$ can be factorized according to

$$U_0 \xrightarrow{(D^2 f, Vf)} L_\alpha^2(E, F) \times L_\alpha(F, E) \xrightarrow{h} L_\alpha(E \times F, E),$$

where h is defined by $h(\phi, \psi) = -\psi \circ \phi \circ (\text{id}_E, \psi)$ for $\phi = D^2 f(x)$ and $\psi = Vf(x)$. By theorem 0.3.5 in [2] h is continuous for $\alpha = c$. Since the category K_α is closed under finite products and $?^\alpha$ is a coreflector it follows that h is continuous. Thus it is true for $p = 1$. Since h is bilinear it is differentiable of class C_α^∞ , and consequently the map Vf is differentiable of class C_α^∞ by induction. Thus the theorem is proved.

We shall now consider examples of coreflective subcategories of LC which are closed under finite products and the coreflectors $?^\alpha$ fulfill the assumption that the identity mapping $C_\alpha(U, F) \rightarrow C_c(U, F)$ is continuous.

EXAMPLE 1. Let K_α be the category $K_c = LC$; $?^\alpha$ is the identity functor $1_{LC} = ?^c$.

EXAMPLE 2. Let K_α be the category K_e of equable locally convex limit vector spaces [2]. The coreflector $?^e : LC \rightarrow K_e$ is the identity on morphisms and on objects E it is characterized as follows: a filter F on E converges to zero in E^e iff $WG = G \leq F$ for some filter G which converges to zero in E .

EXAMPLE 3. Let K be the category K_M of Marinescu spaces [2]. The coreflector $?^M : LC \rightarrow K_M$ is the identity on morphisms and on objects E it is characterized as follows: a filter F on E converges to zero in E^M iff $\forall G = G \leq F$ and $\bigcap \{KG : G \in G\} \in G$ for some filter G which converges to zero in E .

EXAMPLE 4. Let K_α be the category K_b of bornological locally convex limit vector spaces. The coreflector $?^b : LC \rightarrow K_b$ is the identity on morphisms and on objects E it is characterized as follows: a filter F on E converges to zero in E^b iff $\forall B \leq F$ for some bounded subset $B \subseteq E$, i.e. some set B such that $\forall B$ converges to zero in E .

Example 1 gives us the inverse function theorem of Nash and Moser by Hamilton [1]. From example 3 we derive the inverse function theorem of Nash and Moser for the differentiability of class C_M^∞ (C_Δ^∞ in Keller [2]). In [4] Kriegl has discussed smooth mappings between locally convex spaces, where a mapping is called smooth iff its composition with smooth curves are smooth. He has compared this concept of smoothness with different C_α^∞ -differentiabilities (see [2]). From [2] and [4] follow that a mapping between Fréchet spaces is smooth iff it is C_c^∞ -differentiable. Thus the inverse function theorem of Nash and Moser is valid for this concept of smoothness.

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