

RESEARCH NOTES

AN INTEGRAL EQUATION ASSOCIATED WITH
 LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

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(Received March 7, 1985)

ABSTRACT. Associated with each linear homogeneous differential equation

$y^{(n)} = \sum_{i=0}^{n-1} a_i(x)y^{(i)}$ of order n on the real line, there is an equivalent integral equation

$$f(x) = f(x_0) + \int_{x_0}^x h(u)du + \int_{x_0}^x \left[\int_{x_0}^u G_{n-1}(u,v)a_0(v)f(v)dv \right] du$$

which is satisfied by each solution $f(x)$ of the differential equation.

KEY WORDS AND PHRASES. *Linear homogeneous differential equations, Integral equations, Initial value problems, Variation of parameters formula, Uniform convergence.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 34A

1. INTRODUCTION.

Let n be a positive integer, I be an interval on the real line R and $C(I)$ be the class of all functions continuous on I . Let

$$y^{(n)} = a_{n-1}(x)y^{(n-1)} + \dots + a_0(x)y, \quad x \in I \tag{1.1}$$

be any n -th order (normalized) ordinary linear homogeneous differential equation, where $a_i(x) \in C(I)$, $i=0,1,2,\dots,(n-1)$.

The purpose of this article is to derive an equivalent integral equation satisfied by the solutions of the linear homogeneous differential equation (1.1).

2. MAIN RESULTS.

THEOREM. Let $f(x)$ be a solution of (1.1) defined on I and $x_0 \in I$. Then $f(x)$ is also a solution of the integral equation

$$f(x) = f(x_0) + \int_{x_0}^x h(u)du + \int_{x_0}^x \left[\int_{x_0}^u G_{n-1}(u,v)a_0(v)f(v)dv \right] du, \tag{2.1}$$

where $h(x)$ is the unique solution of the $(n-1)$ -th order linear homogeneous differential equation

$$y^{(n-1)} = a_{n-1}(x)y^{(n-2)} + \dots + a_1(x)y, \quad x \in I \tag{2.2}$$

$x_0 \in [a, b]$.

Let $M = \max |G_{n-1}(u, v)a_0(v)|$, $s = \max |f_1(v)|$ for each $u, v \in [a, b]$. One can now see very easily that for each $x \in [a, b]$,

$$|f_{k+1}(x) - f_k(x)| \leq M^k s^k (b-a)^{2k} / 2k!$$

$$|f'_{k+1}(x) - f'_k(x)| \leq M^k s^k (b-a)^k / k!,$$

using recursively the bounds for $|f_{i+1}(x) - f_i(x)|$, $i = 1, 2, 3, \dots$. Since each of the series $\sum M^k s^k (b-a)^{2k} / 2k!$, $\sum M^k s^k (b-a)^k / k!$ converges, we conclude by Weierstrass' M-test that each of the series of functions

$$f_1(x) + \sum_{k=1}^{\infty} (f_{k+1}(x) - f_k(x)), \quad f'_1(x) + \sum_{k=1}^{\infty} (f'_{k+1}(x) - f'_k(x))$$

converges uniformly on $[a, b]$ and hence on B . Therefore there is a function $g(x) \in C^1(I)$ such that

$$f_1(x) + \sum_{k=1}^{\infty} (f_{k+1}(x) - f_k(x)) = \lim_{k \rightarrow \infty} f_k(x) = g(x)$$

$$f'_1(x) + \sum_{k=1}^{\infty} (f'_{k+1}(x) - f'_k(x)) = \lim_{k \rightarrow \infty} f'_k(x) = g'(x)$$

for all $x \in I$. In particular $g(x_0) = f(x_0)$.

Also from (2.5) we get by taking limit as $k \rightarrow \infty$

$$g'(x) = h(x) + \int_{x_0}^x G_{n-1}(x, u) a_0(u) g(u) du, \quad x \in I. \tag{2.7}$$

Hence

$$g(x) = f(x_0) + \int_{x_0}^x h(u) du + \int_{x_0}^x \left[\int_{x_0}^u G_{n-1}(u, v) a_0(v) g(v) dv \right] du \tag{2.8}$$

Again, relation (2.7) implies by (2.3) that $g'(x)$ is the unique solution of the initial value problem

$$y^{(n-1)} = a_{n-1}(x)y^{(n-2)} + \dots + a_1(x)y + a_0(x)g(x)$$

$$y'(x_0) = f'(x_0), \dots, y^{(n-1)}(x_0) = f^{(n-1)}(x_0).$$

Therefore $g(x) \in C^n(I)$ and

$$g^{(n)}(x) = a_{n-1}(x)g^{(n-1)}(x) + \dots + a_0(x)g(x), \quad x \in I.$$

In other words, $g(x)$ is the unique solution of the homogeneous initial value problem

$$y^{(n)} = a_{n-1}(x)y^{(n-1)}(x) + \dots + a_0(x)y, \quad x \in I.$$

$$y(x_0) = f(x_0), \quad y'(x_0) = f'(x_0), \dots, y^{(n-1)}(x_0) = f^{(n-1)}(x_0).$$

Hence $f(x) = g(x)$ for all $x \in I$. Therefore, by (2.8)

$$f(x) = f(x_0) + \int_{x_0}^x h(u) du + \int_{x_0}^x \left[\int_{x_0}^u G_{n-1}(u, v) a_0(v) f(v) dv \right] du.$$

This completes the proof.

REMARK. The above proof clearly shows how a solution of a linear homogeneous equation with prescribed initial values can be constructed out of a solution $h(x)$ and the Green's Function $G_{n-1}(x,u)$ of a lower order homogeneous linear equation. This is specially significant in case of second order homogeneous equations, as solutions $\{ce^{A(x)}\}$ and the Green's function $G_1(x,u) = e^{A(x)-A(u)}$,

$[A(x) = \int_{x_0}^x a_1(u)du]$, of first order homogeneous equation $y' = a_1(x)y$ are readily

available.

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