

A DIGRAPH EQUATION FOR HOMOMORPHIC IMAGES

ROBERT D. GIRSE and RICHARD A. GILLMAN

Department of Mathematics
Idaho State University
Pocatello, Idaho 83209

(Received November 20, 1985)

ABSTRACT. The definitions of a homomorphism and a contraction of a graph are generalized to digraphs. Solutions are given to the graph equation $\overline{\phi(D)} = \theta_\phi(\overline{D})$.

KEY WORDS AND PHRASES. Homomorphisms of graphs, contractions of graphs, digraphs.
1980 AMS SUBJECT CLASSIFICATION CODE. 05C20

By a graph G we mean a finite graph with no multiple edges or loops. If graphs G and H are isomorphic we write $G = H$. An elementary homomorphism of a graph G is an identification of two non-adjacent vertices of G and a homomorphism is a sequence of elementary homomorphisms. A homomorphism of G onto H preserves adjacency. Likewise, an elementary contraction of G is the identification of two adjacent vertices of G and a contraction is a sequence of elementary contractions [1]. Thus for every homomorphism ϕ of G there is a related contraction θ_ϕ of the complement of G , \overline{G} . This contraction is constructed as follows: ϕ is a sequence of elementary homomorphisms $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ so we let θ_ϕ be sequence of elementary contractions $\theta_1, \theta_2, \dots, \theta_n$ where θ_i identifies the same vertices in \overline{G} that ϵ_i identifies in G .

Recently [2] the graph equation $\overline{\phi(G)} = \theta_\phi(\overline{G})$ was studied. In this paper, we generalize the definition of a homomorphism and its related contraction to digraphs and find general solutions to this graph equation. In doing so, we find an easier proof of the result given in [2].

A digraph D consists of a finite vertex set $V(D)$ together with a set $E(D)$ of ordered pairs of distinct elements of $V(D)$, called arcs. Again, if D_1 is isomorphic to D_2 we write $D_1 = D_2$. By an elementary homomorphism of D we mean an identification of two mutually non-adjacent vertices of D (neither uv nor vu are in $E(D)$). Similarly, an elementary contraction is an identification of two mutually adjacent vertices of D (both uv and vu are in $E(D)$). A homomorphism (contraction) of D is again a sequence of elementary homomorphisms (contractions). The contraction θ_ϕ of \overline{D} related to the homomorphism ϕ of D is defined as for undirected graphs.

We will use the following notation as need arises: $Ib(u)$ is the set of vertices v of D such that vu is an arc of D , $Ob(u)$ is the set of vertices v of D such that uv is an arc of D , and $A(u)$ is the adjacency set of u in the graph G .

THEOREM 1. Let ϵ be an elementary homomorphism of D identifying vertices u_1 and u_2 . Then $\epsilon(D) = \theta_\epsilon(\bar{D})$ if and only if $Ib(u_1) = Ib(u_2)$ and $Ob(u_1) = Ob(u_2)$.

PROOF. Let $u = \epsilon(u_1) = \theta_\epsilon(u_1)$. First suppose that $Ob(u_1) \neq Ob(u_2)$. Excluding u as a possible endpoint of an arc, we have vv' is an arc of $\overline{\epsilon(D)}$ if and only if vv' is an arc of $\theta_\epsilon(\bar{D})$. Hence there is a one to one correspondence of those arcs in $\overline{\epsilon(D)}$ without u as an endpoint and those of $\theta_\epsilon(\bar{D})$ without u as an endpoint. The vertex v of the arc uv must be in $Ob(u_1) \cap Ob(u_2)$, $(Ob(u_1) \cup Ob(u_2))^c$, or $Ob(u_1) \nabla Ob(u_2)$, the symmetric difference. In the first case, uv is not an arc of $\overline{\epsilon(D)}$ or $\theta_\epsilon(\bar{D})$ and in the second case, uv is an arc of both. The latter case implies that uv is not an arc of $\overline{\epsilon(D)}$ but is an arc of $\theta_\epsilon(\bar{D})$. Thus for every vertex in $Ob(u_1) \nabla Ob(u_2)$, $\theta_\epsilon(\bar{D})$ has one more arc than $\overline{\epsilon(D)}$. The same holds for vertices in $Ib(u_1) \nabla Ib(u_2)$. Thus if $Ob(u_1) \neq Ob(u_2)$ or $Ib(u_1) \neq Ib(u_2)$, $|E(\theta_\epsilon(\bar{D}))| > |E(\overline{\epsilon(D)})|$ and hence $\overline{\epsilon(D)} \neq \theta_\epsilon(\bar{D})$. Now let $Ib(u_1) = Ib(u_2)$ and $Ob(u_1) = Ob(u_2)$. We will use the identity map from $V(\overline{\epsilon(D)})$ onto $V(\theta_\epsilon(\bar{D}))$ and hence need only consider arcs to and from u . If uv is in $E(\theta_\epsilon(\bar{D}))$ then u_1v and u_2v are arcs in \bar{D} . Thus u_1v and u_2v are not arc of D and subsequently uv is in $E(\overline{\epsilon(D)})$. By the same argument, if uv is an arc of $\overline{\epsilon(D)}$, uv will be an arc of $\theta_\epsilon(\bar{D})$. This holds for arcs vu , so $\overline{\epsilon(D)} = \theta_\epsilon(\bar{D})$.

COROLLARY 1: $\overline{\phi(D)} = \theta_\phi(\bar{D})$ if and only if ϕ is a sequence of elementary homomorphisms, each of which satisfies the conditions of Theorem 1.

A digraph D is pseudo-complete n -partite if there is a partition V_1, V_2, \dots, V_n such that u, u' in V_i for some i implies u and u' are mutually non-adjacent, if u is an element of V_i and v is an element of V_j , $i \neq j$, then either uv or vu is an arc of D , and finally if u and u' are in V_i , v and v' are in V_j , $i \neq j$, and uv is an arc then uv' , $u'v$, and $u'v'$ are also.

THEOREM 2. $\overline{\phi(D)} = \theta_\phi(\bar{D})$ for all homomorphisms ϕ of D if and only if D is pseudo-complete n -partite.

PROOF. If D is pseudo-completely n -partite, every elementary homomorphism identifies two vertices u_1 and u_2 in the same partition set and thus $Ib(u_1) = Ib(u_2)$ and $Ob(u_1) = Ob(u_2)$. Hence $\overline{\epsilon(D)} = \theta_\epsilon(\bar{D})$ for every elementary homomorphism and thus for every homomorphism of D . Conversely, partition $V(D)$ according to the relation: u_1 and u_2 are in V_i if and only if $Ib(u_1) = Ib(u_2)$ and $Ob(u_1) = Ob(u_2)$. We need only show that if u_1 is in V_i and u_2 is in V_j , $i \neq j$, then either u_1u_2 or u_2u_1 is in $E(D)$. Suppose u_1 and u_2 are mutually non-adjacent and let ϵ be the elementary homomorphism identifying them. Since $\overline{\epsilon(D)} = \theta_\epsilon(\bar{D})$, $Ob(u_1) = Ob(u_2)$ and $Ib(u_1) = Ib(u_2)$ by Theorem 1 and hence u_1 and u_2 are in the same partition set. Thus if u_1 is in V_i and u_2 is in V_j , $i \neq j$, there is an arc between them and D must be pseudo-complete n -partite.

If, for every vertex u of D , $Ib(u) = Ob(u)$, D is a symmetric digraph and can be represented by a graph G . This leads to the following corollaries to Theorems 1 and 2.

COROLLARY 2. An elementary homomorphism ϵ identifying vertices u and v of a graph G satisfies $\overline{\epsilon(G)} = \theta_\epsilon(\overline{G})$ if and only if $A(u_1) = A(u_2)$.

COROLLARY 3. A homomorphism ϕ of G satisfies $\overline{\phi(G)} = \theta_\phi(\overline{G})$ if and only if ϕ is a sequence of elementary homomorphisms, each satisfying Corollary 2.

COROLLARY 4. $\overline{\phi(G)} = \theta_\phi(\overline{G})$ for every homomorphism ϕ of G if and only if G is complete n -partite.

A study of the equation $\phi(D) = \theta_\phi(\overline{D})$ would be interesting, yet is apparently difficult considering the work done in [2] for graphs. We conjecture that if $D = \overline{D}$ and $\phi(D) = \theta_\phi(\overline{D})$, ϕ nontrivial, then D is a symmetric digraph.

REFERENCES

1. HARARY, F. Graph Theory, Addison-Wesley, 1969.
2. GIRSE, R. D. Homomorphisms of Complete n -Partite Graphs, Internat. J. Math Math. Sci., 9(1986), 193-196.
3. GIRSE, R. D. and GILLMAN, R. A. Homomorphisms and Related Contractions of Graphs, Internat. J. Math. Math. Sci., submitted.

Special Issue on Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/bvp/guidelines.html>. Authors should follow the Boundary Value Problems manuscript format described at the journal site <http://www.hindawi.com/journals/bvp/>. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	May 1, 2009
First Round of Reviews	August 1, 2009
Publication Date	November 1, 2009

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de

Compostela, Santiago de Compostela 15782, Spain;
juanjose.nieto.roig@usc.es

Guest Editor

Donal O'Regan, Department of Mathematics, National University of Ireland, Galway, Ireland;
donal.oregan@nuigalway.ie