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ON THE CHARACTERISTIC FUNCTION OF A SUM OF M-DEPENDENT RANDOM VARIABLES

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ABSTRACT. Let $S = f_1 + f_2 + \ldots + f_n$ be a sum of 1-dependent random variables of zero mean. Let $\sigma^2 = E\,S^2$, $L = \sigma^{-3} \sum_{1\le i\le n} E|f_i|^3$. There is a universal constant a such that for $a|t|L < 1$, we have

$$|E \exp(itS\sigma^{-1})| \le (1+a|t|)\sup\{(a|t|L)^{-1/4 \ln L}, \exp(-t^2/80)\}.$$

This bound is a very useful tool in proving Berry-Esseen theorems.

KEY WORDS AND PHRASES. Characteristic Function, m-dependent random variable, Berry-Esseen bound.
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1. INTRODUCTION.

Consider a sequence of independent random variables $f_1, f_2, \ldots, f_n$ of zero means having third moments. Let $S = f_1 + \ldots + f_n$ and $\sigma^2 = ES^2$.

If $t \le \sigma E(f_i^2)/E|f_i|^3$ for each $i \le n$, one has

$$|E[\exp(itS\sigma^{-1})]| \le \prod_{i\le n} |E[\exp(itf_i\sigma^{-1})]| \le \prod_{i\le n} \exp(-t^2/3\; Ef_i^2\sigma^{-2})$$
$$\le \exp(-t^2/3).$$

This trivial estimate plays a fundamental role in the proof of Berry-Esseen rates of convergence in the central limit theorem. The purpose of this work is to find an estimate of $|E[\exp(itS\sigma^{-1})]|$ for the sequence of m-dependent random variables.

We say that a sequence $(f_i)_{i=1}^n$ of random variables is m-dependent if for each $1 \le p \le n-m-1$, the sequences $(f_i)_{i\le p}$ and $(f_i)_{i>p+m}$ are independent of each other.

In a recent very interesting paper by V. V. Shergin[1], the author gives the best rate of convergence in the central limit theorem for m-dependent random variables. We will estimate the bound of $|E[\exp(itS\sigma^{-1})]|$ by Shergin's methods. This result extracts the most important ideas of Shergin's work. Also we want to point out that this estimate turns out to be an essential tool in the proof of Berry-Esseen type bounds in other limit theorems for m-dependent random variables. In a subsequent work, we shall establish such a convergence rate for U-statistics[2] and an Edgeworth expansion for a sum of m-dependent random variables[3].

2. CONSTRUCTION.

We follow the lines of Shergin's ingenious construction to decompose S in an amenable way. We do not however assume the reader to be familiar with Shergin's paper. The exposition is self contained, and some long details of his proof are eliminated by our approach.

We assume now on m=1. We denote $a_0, a_1, \ldots, a_\ell$ universal constants. No attempt is made at finding optimal values for these universal constants, since the numerical values involved here are too large to be of any interest.

Set $U = \sum_{i\le n} E|f_i|^3$, $L = U\sigma^{-3}$ and $R = -\ln L$. In the sequel, we assume $R \ge 10$. It follows that for $i \le n$, we have

$$Ef_i^2 \le (E|f_i|^3)^{2/3} \le \sigma^2 L^{2/3} \le \sigma^2/50R. \tag{2.1}$$

By induction we define indices s(i) as follows. Set s(1) = 1, and

$$s(i+1) = 1+\min\{s\colon s>s(i),\ E(f_{s(i)}+\ldots+f_s)^2 \ge \sigma^2/R\}.$$

The construction stops at an index h such that either s(h) = n or

$E(f_{s(h)}+\ldots+f_s)^2 < \sigma^2/R$ for $s(h)<s\le n$.

LEMMA 1. ([1]) We have $10R/11 \le h \le 2R$ and $s(i+1) - s(i) \ge 15R$ for $1 \le i \le h-1$.

PROOF. From the 1-dependence of the f_i it follows that

$$\sigma^2 = \sum_{i=1}^{h-1} E(f_{s(i)}+\ldots+f_{s(i+1)-1})^2 + E(f_{s(h)}+\ldots+f_n)^2 + 2\sum_{i=1}^{h-1} E\{f_{s(i+1)-1}f_{s(i+1)}\}.$$

It follows from Schwartz's inequality and (2.1) that

$$\sigma^2 \ge (h-1)\sigma^2/R - 2(h-1)\sup E|f_i|^2 \ge 24\sigma^2(h-1)/25R$$

so that $h \le 25R/24 + 1 \le 2R$. Moreover

$$\sigma^2 = \sum_{i=1}^{h-1} E(f_{s(i)}+\ldots+f_{s(i+1)-2})^2 + E(f_{s(h)}+\ldots+f_n)^2$$

$$+ \sum_{i=1}^{h-1} Ef^2_{s(i+1)-1} + 2\sum_{i=1}^{h-1} E(f_{s(i+1)-2}f_{s(i+1)-1}) + 2\sum_{i=1}^{h-1} E(f_{s(i+1)-1}f_{s(i+1)}).$$

So,

$$\sigma^2 \leq h\sigma^2/R + 5h \sup Ef_i^2 \leq 11h\sigma^2/10R$$

and hence $h \geq 10R/11$. On the other hand, for $i \leq h-1$,

$$\sigma^2/R \leq E(f_{s(i)}+\dots+f_{s(i+1)-1})^2 = \sum_{j=s(i)}^{s(i+1)-1} Ef_j^2 + \sum_{j=s(i)}^{s(i+1)-2} 2E(f_j f_{j+1})$$

$$\leq 3(s(i+1) - s(i)) \operatorname{Max} Ef_i^2 \leq 3(s(i+1) - s(i))\sigma^2/50R$$

which proves the lemma. Q.E.D.

For $i \leq h-1$, let $\tau_i = \sum_{j=s(i)}^{s(i+1)-1} E|f_j|^3$. We have $\sum_{i=1}^{h-1} \tau_i \leq U$. Hence if p is the number of indices $i \leq h-1$ such that $\tau_i \geq 10U(h-1)^{-1}$, we have $p \leq (h-1)/10$. It follows that there are at least $9(h-1)/10$ indices i for which $\tau_i \leq 10U(h-1)^{-1}$.

Let $H = [9(h-1)/20 - 1]$. If $R \geq 10$, we have $H \geq R/10$. This follows from the fact that $h \geq 10R/11$ and straightforward computations. We can moreover select indices $i_1,\dots,i_H$ such that for $1 \leq \ell \leq h$,

$$\tau_{i_\ell} \leq 10U(h-1)^{-1} \leq 20U/R;\ i_{\ell+1} - i_\ell \geq 2,\ 2 \leq i_\ell \leq h-2. \tag{2.2}$$

For $1 \leq \ell \leq H$, we set $s(i_\ell) = a_\ell$, $s(i_\ell+1) = a'_\ell$. We have $a_\ell - a'_\ell \geq 15R \geq 15H$.

Let

$$\overline{f}_\ell = (a'_\ell - a_\ell)^{-1} \sum_{a_\ell \leq j < a'_\ell} E|f_j|.$$

Since there are at least $15H/2 \geq 7H$ indices $a_\ell \leq j < a'_\ell$ for which $E|f_j| \leq 2\overline{f}_\ell$, one can pick indices $p(\ell,-H),\dots,p(\ell,0),\dots,p(\ell,H)$ of $[a_\ell, a'_\ell[$ with this property such that no two of them are consecutive.

LEMMA 2. For each $-H \leq i \leq H$, we have $E|f_{p(\ell,i)}| \leq 40L$.

PROOF. By Holder's inequality, we have,

$$\sum_{a_\ell \leq j < a'_\ell} E|f_j| \leq \sum_{a_\ell \leq j < a'_\ell} (E|f_j|^3)^{1/3} \leq (a'_\ell - a_\ell)^{2/3} \tau_{i_\ell}^{1/3},$$

$$\sum_{a_\ell \leq j < a'_\ell} E|f_j|^3 \leq \sum_{a_\ell \leq j < a'_\ell} (E|f_j|^3)^{2/3} \leq (a'_\ell - a_\ell)^{1/3} \tau_{i_\ell}^{2/3}.$$

As already shown, $E(\sum_{a_\ell \leq j < a'_\ell} f_j)^2 \leq 5 \sum_{a_\ell \leq j < a'_\ell} Ef_j^2$, so we get by combining the above inequalities, and since $\sigma^2 \leq R\, E(\sum_{a_\ell \leq j < a'_\ell} f_j)^2$, $\sigma^2 \overline{f}_\ell \leq 2R\, \tau_{i_\ell} \leq 40U$. Q.E.D.

Set

$$Z_{\ell,0} = f_{p(\ell,0)},\ Z_{2q} = f_{p(\ell,-q)} + f_{p(\ell,q)}, \quad \text{for } 0 \le q \le H,$$

$$Z_{2q+1} = \sum_{p(\ell,-q-1)<i<p(\ell,-q)} f_i + \sum_{p(\ell,q)<i<p(\ell,q+1)} f_i, \quad \text{for } 0 \le q \le H.$$

For $1 \le s \le H$, let $b_s = (e_1,\dots,e_s)$ be a collection of integers $0 \le e_s \le 2H+1$. We set

$$W(b_s) = \sigma^{-1}(S - \sum_{\ell=1} \sum_{j \le e_s} Z_{\ell,j}).$$

For $s<H$, $\ell \le H_1$, $b_s = (e_1,\dots,e_s)$, we set $W(b_s,\ell) = W((e_1,\dots,e_s),\ell)$. Let

$$\phi_s(t) = \max|E[\exp(it\text{-}W(b_s))]|$$

and if $s<H$, let

$$\lambda_s(t) = \max|E[\exp(itW(b_s)) - \exp(itW(b_s,0))]|.$$

Here the maximum is taken over all possible choices of b_s.

3. ESTIMATES.

LEMMA 3. There is a universal constant a_1 such that for $a_1|t|L<1$ and $1 \le s<H$ we have

$$\lambda_s(t) \le (a_1|t|L)^{H+1} + a_1|t|L\phi_{s+1}(t).$$

PROOF. Let us fix b_s, and for $0 \le k \le 2H+1$, let $\gamma_k = \exp(itZ_{s,k}\sigma^{-1})-1$. one sees by induction that

$$\exp(itW(b_s)) - \exp(itW(b_s,0)) = \sum_{\ell=1}^{2H+1} \exp(itW(b_s,\ell)) \prod_{k=0}^{\ell-1} \gamma_k \tag{3.1}$$

$$+ \exp(itW(b_\ell,2H+1)) \prod_{k=0}^{2H+1} \gamma_k.$$

By a well known estimate and lemma 2, we have

$$E|\gamma_{2r}| \le |t|\sigma^{-1}E|Z_r| \le 80|t|L \quad \text{for } 0 \le 2r \le H-1.$$

So, since $|\gamma_k| \le 2$, if we set $\bar{\ell} = [\ell/2]$, we have

$$E|\prod_{k=0}^{\ell-1} \gamma_k| \le 2^{\ell} E \prod_{i \le \bar{\ell}} |\gamma_{2i}| \le (160|t|L)^{\bar{\ell}+1}.$$

Thus the lemma follows (with $a_1 = 320$) by taking expectation in (3.1), and since

$$\sum_{\ell=0}^{\infty} (160|t|L)^{\bar{\ell}+1} \le 2 \text{ for } a_1|t|L<1.$$

LEMMA 4. $\phi_H(t) \le (\exp(-t^2/4R) + a_4|t|L)^H$.

PROOF. We fix $b_H = (e_1,\dots,e_H)$. For $1 \le \ell \le H$, we set

$$r_\ell = p(\ell,-q)-1,\ r'_\ell = p(\ell,q)+1 \text{ if } e_\ell \text{ is of the form } 2q$$

and

$r_\ell = p(\ell,-q-1)$, $r'_\ell = p(\ell,q-1)$ if e_ℓ is of the form $2q+1$.

Let $r'_0 = 1$ and $r_{H+1} = n$. For $0 \le \ell \le H$, let

$$T_\ell = \sum_{r'_\ell \le i \le r_{\ell+1}} f_i .$$

We have $W(b_H) = \sigma^{-1}(T_0+\ldots+T_H)$, and the T_ℓ are independent. Moreover it follows from (2.2) that each T_ℓ is the sum of the f_i over an interval which contains an interval of the type $[s(j_\ell), s(j_\ell+1)[$. It follows that

$$\sigma_\ell^2 = E|T_\ell|^2 \ge E\Big(\sum_{s(j_\ell)\le i<s(j_\ell+1)} f_i\Big)^2 - 2E(f_{s(j_\ell)}f_{s(j_\ell-1)}) - 2E(f_{s(j_\ell+1)}f_{s(j_\ell+2)})$$

$$\ge \sigma^2/2R.$$

Let $\omega_\ell = \sum_{r'_\ell \le i \le r_{\ell+1}} E(f_i)^3$. It follows from the theorem of R. V. Erickson [4] that for each z, $|E(\exp izT_\ell\sigma_\ell^{-1}) - \exp(-z^2/2)| \le a_3|z|\omega_\ell\sigma_\ell^{-3}$.

By taking $z = t\sigma_\ell\sigma^{-1}$ and using $\sigma_\ell^2 \ge \sigma^2/2R$, one gets

$$|E(\exp itT_\ell\sigma^{-1})| \le \exp(-t^2/4R) + 2a_3\sigma^{-3}R\omega_\ell.$$

Thus, we get

$$|E(\exp itW(b_H)| \le \prod_{\ell=q}^{H} (\exp(-t^2/4R) + 2a_3\sigma^{-3}R\omega_\ell).$$

The concavity of the function ln (1+x), and the fact that $\sum_{\ell=0}^{H} \sigma^{-3}R\omega_\ell \le RL \le 10HL$ prove the result. Q.E.D.

4. RESULTS.

PROOSITION 5. If $a_5|t|L \le 1$ (and $L \le e^{-10}$), we have

$$|E(\exp itS\sigma^{-1})| \le (1+a_5|t|)(\exp(-t^2/4R) + a_5|t|L)^H. \tag{4.1}$$

PROOF. Since $\phi_s \le \phi_{s+1} + \lambda_{s+1}$, it follows easily from lemma 3 and by induction that

$$\sum_{\ell=1}^{q} \lambda_\ell(t) \le q(1+a_1|t|L)^q((a_1|t|L)^{H+1} + a_1|t|\phi_{s+1}(t)).$$

So,

$$|E(\exp itS\sigma^{-1})| \le \phi_H + \sum_{s=1}^{H} \lambda_s$$

$$\le \phi_H(t)(1 + Ha_1|t|L(1+a_1|t|L)^H) + H(1+a_1|t|L)^H(a_1|t|L)^H(a_1|t|L)^{H+1}.$$

If $a_1|t|L \le e^{1/2}-1$, we have, since $H \le R = \ln L^{-1}$,

$$H(1+a_1|t|L)^H \le 2(\ln L^{-1})L^{-1/2} \le a_\sigma L^{-1}.$$

So proposition 5 follows from lemma 4 with $a_5 = \sup(10a_1, 2a_3, a_\sigma)$.

It is well worthwhile to reformulate the above result to show more precisely the behaviour of the bound.

THEOREM 6. There exists universal constants a_7 and a_8 such that for $q \in \mathbb{N}$ and $|t|$, $a_7 \le |t|$ and $a_8|t|L \le 1$, and

$$|E(\exp itS\sigma^{-1})| \le (1+a_5|t|)\sup(\exp(-t^2/80), (a_8|t|L)^{\ln L}). \qquad (4.2)$$

PROOF. Let $a_8 = 3a_5$. By taking a_7 large enough, the existence of one $|t|$ satisfying the hypothesis implies $L \ge e^{10}$, so we can assume that (4.1) holds. We can also assume that $a_7 \ge 80a_5$. If $|t| \le 2\sqrt{R}$, we have $\exp(-t^2/4R) \ge 1/e$. Thus, since $H \ge R/10$,

$$(\exp(-t^2/4R) + a_5|t|L)^H \le \exp(-t^2/40)(1+ea|t|L)^H$$
$$\le \exp(-t^2/40 + ea_5|t|LH).$$

Since $LH \le LR \le 1/e$ and $|t| \ge 80a_5$, we have $-t^2/40 + ea_5|t|LH \le -t^2/80$, which proves (4.2) in that case. If $t \ge 2\sqrt{R}$, then $\exp(-t^2/8R) \ge \sqrt{e}(\exp(-t^2/4R))$, and it is easy to check that

$$\exp(-t^2/4R) + a_5|t|L \le \text{Max}\{\exp(-t^2/8R), 3a_5|t|L\}$$

and theorem 6 follows. Q.E.D.

REMARK. (1) In case of a m-dependent (m>1) sequence of random variables, an estimate of $|E(\exp itS\sigma^{-1})|$ can be obtained by considering S as the sum of 1-dependent blocks of f_i.

(2) The constant 1/4 in the exponent of (4.2) plays no particular role. It is clear from the method that it can be replaced by any number; but the values of a_5 and a_8 depend on this exponent. However for the applications we have in mind, any positive number will be sufficient.

To support our claim that theorem 6 is useful tool, we deduce Shergin's theorem in a simpler way. Let Φ be the distribution function with the normal law.

SHERGIN'S THEOREM.

$$\sup_t |P(S<t) - \Phi(t)| \le aL.$$

PROOF. It is possible either to use the construction of sections 2 and 3 or to do again a similar but much simpler construction. In order not to repeat arguments already used, we choose the first approach. Let $q = [H/2]$ and $p = p(q, 0)$. Let $S_1 = \sum_{0\le i<p} f_i$ and $S_2 = \sum_{p<i\le n} f_i$. For $s(i_\ell)\le s \le p$ (resp. $p < s \le s(i_{\ell+1})$), it is easily seen that $E(\sum_{1\le i\le s} f_i)^2 \ge \sigma^2/10$ (resp. $E(\sum_{1\le i\le n} f_i)^2 \ge \sigma^2/16$). The method of lemma 3 and the result of theorem 6 gives for $4a_8|t|L \le 1$:

$$|E(\exp itS\sigma^{-1}) - E(\exp it(S_1 + S_2)\sigma^{-1}|$$

$$\le (a_1|t|L)^{H+1} + a_1|t|L(1+4a_5|t|)\mathrm{Max}\{\exp(-t^2/320),\ (4a_8|t|L)^{-\frac{1}{4}\ln L/64}\}.$$

Moreover, if $\sigma_1^2 = E\,S_1^2$ and $\sigma_2^2 = E\,S_2^2$, we get for $\varepsilon = 1, 2$, from Erickson's theorem:

$$|E(\exp itS_\varepsilon\sigma^{-1}) - \exp(-t^2\sigma_\varepsilon^2/2\sigma^2)| \le 16a_3|t|L.$$

So it follows, using again theorem 6, that

$$|E(\exp it(S_1 + S_2)\sigma^{-1}) - \exp(-t^2(\sigma_1^2 + \sigma_2^2)/2\sigma^2)|$$

$$\le 36a_3|t|L(1+4a_5|t|)\mathrm{Max}\{\exp(-t^2/320),\ (4a_8|t|L)^{-\frac{1}{4}\ln L/64}).$$

Now, if we set $T^{-1} = 4e^8a_8L$, a straightforward computation gives

$$J(T) = \int_{-T}^{T} t^{-1}|E(\exp itS) - \exp(-t^2(\sigma_1^2 + \sigma_2^2)/2\sigma^2)| \le a_{10}L.$$

The familiar Esseen inequality gives

$$\sup_x (P(S\sigma^{-1} < x) - \Phi'(x)) \le a(J(T) + T^{-1}) \le a_{11}L$$

where $\Phi'(x)$ is the normal distribution function with variance $k^2 = \sigma^{-2}(\sigma_1^2 + \sigma_2^2) = \sigma^{-2}E(S - f_p)^2$. We have

$$\sup_x (\Phi'(x) - \Phi(x)) \le 1-k^2 \le \sigma^2(E(f_p^2) + 2E(f_{p-1}f_p) + 2E(f_pf_{p+1})).$$

We can also assume that at the time we picked the indices $p(q,j)$ we have made the extra effort to choose $p = p(q,0)$ such that for $\varepsilon = 1,0,1$, we have $E(f_{p+\varepsilon}^2) \le 10(a_q' - a_q) \sum_{a_\ell\le j\le a_\ell'} E|f_j|^2$. It then follows by an estimate similar to lemma 2 that the right hand side of the parenthesis is also bounded by $a_{12}L$, and concludes the proof.

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