

ON SOME RESULTS FOR λ -SPIRAL FUNCTIONS OF ORDER α

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ABSTRACT. The results of various kinds concerning λ -spiral functions of order α in the unit disk U are given in this paper. They represent mainly the generalizations of the previous results of the authors.

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1. INTRODUCTION.

Let A_n denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}) \quad (1.1)$$

which are regular in the unit disk $U = \{z: |z| < 1\}$.

For a function $f(z)$ belonging to the class A_n we say that $f(z)$ is λ -spiral of order α if and only if

$$\operatorname{Re} \left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} \right\} > \alpha \cos \lambda \quad (1.2)$$

for some real λ ($|\lambda| < \pi/2$), for some α ($0 \leq \alpha < 1$), and for all $z \in U$.

We denote by $S_n^\lambda(\alpha)$ the class of all such functions. In the case $n = 1$ we write A and $S^\lambda(\alpha)$ instead of A_1 and $S_1^\lambda(\alpha)$, respectively. The class $S^\lambda(\alpha)$ has been considered by Libera [1].

We note that $S^\lambda(0) = S^\lambda$, $S_n^0(\alpha) = S_n^*(\alpha)$, and $S^0(0) = S^*$, where S^λ , $S_n^*(\alpha)$, and S^* denote the classes of functions which are λ -spiral, starlike of order α and type (1.1), and starlike, respectively.

Let $f(z)$ and $g(z)$ be regular in the unit disk U . Then a function $f(z)$ is said to be subordinate to $g(z)$ if there exists a regular function $w(z)$ in the unit disk U satisfying $w(0) = 0$ and $|w(z)| < 1$ ($z \in U$) such that $f(z) = g(w(z))$. For this relation the following symbol $f(z) \prec g(z)$ is used. In particular, if $g(z)$ is univalent in the unit disk U the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

2. RESULTS AND CONSEQUENCES.

First we give the following result due to Miller and Mocanu [2].

LEMMA 1. Let $\phi(u,v)$ be a complex valued function,

$$\phi: D \longrightarrow \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies the following conditions:

- (i) $\phi(u,v)$ is continuous in D ;
- (ii) $(1,0) \in D$ and $\text{Re}\{\phi(1,0)\} > 0$;
- (iii) $\text{Re}\{\phi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -n(1 + u_2^2)/2$.

Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ be regular in the unit disk U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$\text{Re}\{\phi(p(z), zp'(z))\} > 0 \quad (z \in U),$$

then

$$\text{Re}\{p(z)\} > 0 \quad (z \in U).$$

By using the above lemma, we prove

THEOREM 1. Let the function $f(z)$ defined by (1.1) be in the class $S_n^\lambda(\alpha)$ and let

$$0 < \beta \leq \frac{n}{2(1 - \alpha) \cos \lambda} \quad (2.1)$$

Then we have

$$\text{Re} \left\{ \left\{ \frac{f(z)}{z} \right\}^{\beta e^{i\lambda}} \right\} > \frac{n}{2\beta(1 - \alpha) \cos \lambda + n} \quad (z \in U).$$

PROOF. If we put

$$B = \frac{n}{2\beta(1 - \alpha) \cos \lambda + n} \quad (2.2)$$

and

$$\left(\frac{f(z)}{z} \right)^{\beta e^{i\lambda}} = (1 - B)p(z) + B, \quad (2.3)$$

where β satisfies (2.1) then $p(z)$ is regular in the unit disk U and $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$. From (2.3) after taking the logarithmical differentiation we have that

$$\beta e^{i\lambda} \frac{zf'(z)}{f(z)} - \beta e^{i\lambda} = (1 - B) \cdot \frac{zp'(z)}{(1 - B)p(z) + B}, \quad (2.4)$$

and from there

$$e^{i\lambda} \frac{zf'(z)}{f(z)} - \alpha \cos \lambda = e^{i\lambda} - \alpha \cos \lambda + \frac{(1 - B)zp'(z)}{\beta\{(1 - B)p(z) + B\}} \quad (2.5)$$

Since $f(z) \in S_n^\lambda(\alpha)$ then from (2.5) we get

$$\operatorname{Re} \left\{ e^{i\lambda} - \alpha \cos \lambda + \frac{(1 - B)zp'(z)}{\beta\{(1 - B)p(z) + B\}} \right\} > 0 \quad (z \in U). \quad (2.6)$$

Let consider the function $\phi(u, v)$ defined by

$$\phi(u, v) = e^{i\lambda} - \alpha \cos \lambda + \frac{(1 - B)v}{\beta\{(1 - B)u + B\}}$$

(it is noted $u = p(z)$ and $v = zp'(z)$). Then $\phi(u, v)$ is continuous in $D = (\mathbb{C} - \{-B/(1-B)\}) \times \mathbb{C}$. Also, $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = (1 - \alpha) \cos \lambda > 0$. Furthermore, for all $(iu_2, v_1) \in D$ such that $v_1 \leq -n(1 + u_2^2)/2$ we have

$$\begin{aligned} \operatorname{Re}\{\phi(iu_2, v_1)\} &= (1 - \alpha) \cos \lambda + \operatorname{Re} \left\{ \frac{(1 - B)v_1}{\beta\{(1 - B)iu_2 + B\}} \right\} \\ &= (1 - \alpha) \cos \lambda + \frac{B(1 - B)v_1}{\beta\{(1 - B)^2 u_2^2 + B^2\}} \\ &\leq (1 - \alpha) \cos \lambda - \frac{nB(1 - B)(1 + u_2^2)}{2\beta\{(1 - B)^2 u_2^2 + B^2\}} \\ &= \frac{(1 - B)\{4\beta^2(1 - \alpha)^2 \cos^2 \lambda - n^2\}u_2^2}{2\beta\{(1 - B)^2 u_2^2 + B^2\}} \\ &\leq 0 \end{aligned}$$

because $0 < B < 1$ and $2\beta(1 - \alpha) \cos \lambda - n \leq 0$. Therefore, the function $\phi(u, v)$ satisfies the conditions in Lemma 1. This proves that $\operatorname{Re}\{p(z)\} > 0$ for $z \in U$, that is, that from (2.3),

$$\operatorname{Re} \left\{ \left\{ \frac{f(z)}{z} \right\}^{\beta e^{i\lambda}} \right\} > B \quad (z \in U),$$

which is equivalent to the statement of Theorem 1.

Taking $\alpha = 0$ and $\beta = n/2 \cos \lambda$ in Theorem 1, we have

COROLLARY 1. Let the function $f(z)$ defined by (1.1) be in the class $S_n^\lambda(0)$. Then

$$\operatorname{Re} \left\{ \left\{ \frac{f(z)}{z} \right\}^{(n/2 \cos \lambda) e^{i\lambda}} \right\} > \frac{1}{2} \quad (z \in U).$$

REMARK 1. If $\lambda = 0$ in Theorem 1, then we have the former result given by the authors in [3]. If $\lambda = 0$ in Corollary 1, then we have the earlier result given by Golusin [4].

THEOREM 2. Let β be a fixed real number, $0 \leq \beta < 1$, and let the function $f(z)$ be in the class $S^\lambda(\alpha)$. Let

$$g(z) = z \left\{ \frac{f(z)}{z} \right\}^\gamma \frac{1}{(1-z)^{2\mu} e^{-i\lambda} \cos \lambda} \quad (z \in U), \quad (2.7)$$

where $0 < \gamma \leq (1 - \beta)/(1 - \alpha)$ and $\mu = 1 - \beta - \gamma(1 - \alpha)$. Then the function $g(z)$ is in the class $S^\lambda(\beta)$.

PROOF. Let β ($0 \leq \beta < 1$) be a given real number. Then from (2.7) by using the logarithmic differentiation and some simple transformations, we have that

$$\begin{aligned} & e^{i\lambda} \frac{zg'(z)}{g(z)} - \beta \cos \lambda \\ &= \gamma \left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} + \alpha \cos \lambda \right\} + (1 - \gamma) e^{i\lambda} - \left\{ \beta - \alpha\gamma - \frac{2\mu z}{1-z} \right\} \cos \lambda \\ &= \gamma \left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} - \alpha \cos \lambda \right\} + \mu \cos \lambda \frac{1+z}{1-z} - i(1 - \gamma) \sin \lambda. \quad (2.8) \end{aligned}$$

From (10) we have

$$\begin{aligned} & \operatorname{Re} \left\{ e^{i\lambda} \frac{zg'(z)}{g(z)} - \beta \cos \lambda \right\} \\ &= \operatorname{Re} \left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} - \alpha \cos \lambda \right\} + \mu \cos \lambda \left\{ \frac{1+z}{1-z} \right\} \\ &> 0 \end{aligned} \quad (z \in U), \quad (2.9)$$

because of the suppositions for γ , $f(z)$ and μ given in the statement of Theorem 2. Thus we complete the proof of Theorem 2.

COROLLARY 2. Let $0 \leq \beta \leq \alpha$, and let $f(z) \in S^\lambda(\alpha)$. Then the function $g(z)$ defined by

$$g(z) = \frac{f(z)}{(1-z)^{2(\alpha-\beta)} e^{-i\lambda} \cos \lambda} \quad (2.10)$$

belongs to the class $S^\lambda(\beta)$.

PROOF. Since $0 \leq \beta \leq \alpha$, it follows that $1 \leq (1-\beta)/(1-\alpha)$ and we may choose $\gamma = 1$ in Theorem 2. Also, we have $\mu = \alpha - \beta$.

REMARK 2. In particular, for $\lambda = 0$ in Corollary 2 we have that if $f(z) \in S^*(\alpha)$ and $0 \leq \beta \leq \alpha$, then the function

$$g(z) = \frac{f(z)}{(1-z)^{2(\alpha-\beta)}}$$

belongs to the class $S^*(\beta)$.

This is the earlier result given by the authors [5].

Letting $\lambda = 0$ in Theorem 2, we have

COROLLARY 3. Let $f(z) \in S^*(\alpha)$ and let $0 \leq \beta < 1$. Then the function $g(z)$ defined by

$$g(z) = z \left\{ \frac{f(z)}{z} \right\}^\gamma \frac{1}{(1-z)^{2\mu}}$$

belongs to the class $S^*(\beta)$, where $0 < \gamma \leq (1 - \beta)/(1 - \alpha)$ and $\mu = 1 - \beta - \gamma(1 - \alpha)$.

In order to derive the following theorem, we shall apply the next result given by Robertson [6].

LEMMA 2. Let the function $f(z)$ belonging to A be univalent in the unit disk U . For $0 \leq t \leq 1$, let $F(z, t)$ be regular in the unit disk U with $F(z, 0) \equiv f(z)$ and $F(0, t) \equiv 0$. Let p be a positive real number for which

$$F(z) = \lim_{t \rightarrow +0} \frac{F(z, t) - F(z, 0)}{zt^p}$$

exists. Further, let $F(z, t)$ be subordinate to $f(z)$ in the unit disk U for $0 \leq t \leq 1$. Then

$$\operatorname{Re} \left\{ \frac{F(z)}{f'(z)} \right\} \leq 0 \quad (z \in U).$$

If, in addition, $F(z)$ is also regular in the unit disk U and $\operatorname{Re}\{F(0)\} \neq 0$, then

$$\operatorname{Re} \left\{ \frac{f'(z)}{F(z)} \right\} < 0 \quad (z \in U). \quad (2.11)$$

Applying the above lemma, we prove

THEOREM 3. Let the function $f(z)$ be in the class A , and let the function $g(z)$ defined by

$$g(z) = \frac{1}{1 - \alpha \cos \lambda} \left\{ f(z) - \alpha \cos \lambda \int_0^z \frac{f(s)}{s} ds \right\} = z + \dots \quad (2.12)$$

be univalent in the unit disk U , where λ is a real number with $|\lambda| < \pi/2$ and $0 \leq \alpha < 1$. If the function $G(z, t)$ defined by

$$G(z, t) = \frac{1}{1 - \alpha \cos \lambda} \left\{ (1 - te^{-i\lambda})f(z) - \alpha \cos \lambda(1 - t^2) \int_0^z \frac{f(s)}{s} ds \right\} \quad (2.13)$$

is subordinate to $g(z)$, that is, $G(z, t) \prec g(z)$, in the unit disk U for fixed α and λ , and for each t ($0 \leq t \leq 1$), then $f(z)$ is in the class $S^\lambda(\alpha)$.

PROOF. It is easy to show that $G(z, 0) \equiv g(z)$ and $G(0, t) \equiv 0$. We choose $p = 1$ and $F(z, t)$ to be the function $G(z, t)$ defined by (2.13) in Lemma 2. Then we have

$$\begin{aligned} G(z) &= \lim_{t \rightarrow +0} \frac{G(z, t) - G(z, 0)}{zt} = \lim_{t \rightarrow +0} \frac{\partial G(z, t) / \partial t}{-e^{-i\lambda} f(z)} \\ &= \frac{1}{(1 - \alpha \cos \lambda) z}. \end{aligned} \quad (2.14)$$

From (2.14) we know that $G(z)$ is regular in the unit disk U and

$$\text{Since } \text{Re}\{G(0)\} = \frac{-\cos \lambda}{1 - \alpha \cos \lambda} \neq 0.$$

$$g'(z) = \frac{1}{1 - \alpha \cos \lambda} \left\{ f'(z) - \alpha \cos \lambda \frac{f(z)}{z} \right\},$$

it follows from (2.11) in Lemma 2 that

$$\text{Re} \left\{ \frac{g'(z)}{G(z)} \right\} < 0 \quad (z \in U),$$

which is equivalent to

$$\text{Re} \left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} - \alpha \cos \lambda \right\} > 0 \quad (z \in U).$$

Consequently, we prove that $f(z)$ is in the class $S^\lambda(\alpha)$.

If we put $\alpha = 0$ in Theorem 3, then we have

COROLLARY 4. Let the function $f(z)$ belonging to the class A be univalent in the unit disk U such that

$$(1 - te^{-i\lambda})f(z) \prec f(z) \quad (z \in U),$$

where λ is real such that $|\lambda| < \pi/2$ and $0 \leq t \leq 1$. Then $f(z)$ is in the class $S^\lambda(0)$. This is the former result due to Robertson [6].

For $\lambda = 0$ in Theorem 3, we have the following result for starlike functions of order α .

COROLLARY 5. Let the function $f(z)$ be in the class A and let the function $g(z)$ defined by

$$g(z) = \frac{1}{1 - \alpha} \left\{ f(z) - \alpha \int_0^z \frac{f(s)}{s} ds \right\} \quad (0 \leq \alpha < 1)$$

be univalent in the unit disk U. If

$$G(z, t) = \frac{1}{1 - \alpha} \left\{ (1 - t)f(z) - \alpha(1 - t^2) \int_0^z \frac{f(s)}{s} ds \right\} \prec g(z)$$

in the unit disk U, then $f(z)$ belongs to the class $S^*(\alpha)$. This is the previous result given by Obradović [7].

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