

FINE TOPOLOGY ON FUNCTION SPACES

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ABSTRACT. This paper studies the topological properties of two kinds of "fine topologies" on the space $C(X,Y)$ of all continuous functions from X into Y .

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1. INTRODUCTION.

The topology of pointwise convergence and the compact-open topology are two of the most commonly used topologies on the set $C(X,Y)$ of continuous functions from a space X into a space Y . These spaces will be denoted by $C_p(X,Y)$ and $C_k(X,Y)$, respectively. If Y is a metric space, the supremum metric topology on $C(X,Y)$ is also commonly used. However, sometimes none of these topologies is strong enough to apply a function space to a given situation, in which case a finer topology may be needed. A good example of this is the use of a "fine topology" on a function space in [4], in which the Baire space property of the function space is used to obtain certain kinds of embeddings into infinite-dimensional manifolds.

This paper studies the topological properties of two kinds of "fine topologies" on $C(X,Y)$. In order to avoid pathologies, all spaces will be Tychonoff spaces. The symbol \mathbb{R} will denote the real line with the usual topology, and \mathbb{R}^+ will denote the positive real line. Also $C(X, \mathbb{R})$ and $C(X, \mathbb{R}^+)$ will be abbreviated as $C(X)$ and $C^+(X)$. Finally let ω denote the set of natural numbers.

1. UNIFORM TOPOLOGIES.

Whenever a space Y has a compatible uniform structure on it, this induces a uniform structure on $C(X,Y)$. If δ is a diagonal uniformity on Y , then for each $D \in \delta$, define

$$\hat{D} = \{(f,g) \in C(X,Y)^2 : \text{for every } x \in X, (f(x),g(x)) \in D\}. \quad (1.1)$$

The family $\{\hat{D} : D \in \delta\}$ is a base for a diagonal uniformity $\hat{\delta}$ on $C(X,Y)$. Denote the resulting topological space by $C_{\hat{\delta}}(X,Y)$. On the other hand, if μ is a covering uniformity on Y , then for each $U \in \mu$, let

$$\hat{U} = \{(f,g) \in C(X,Y)^2 : \text{for every } x \in X, \text{ there exists a } U \in \mu \text{ with } (f(x),g(x)) \in U\}. \quad (1.2)$$

The family $\{\hat{U} : U \in \mu\}$ is also a base for a diagonal uniformity $\hat{\mu}$ on $C(X,Y)$. Denote this space by $C_{\hat{\mu}}(X,Y)$.

There is a natural way of passing from a diagonal uniformity δ to a covering uniformity μ_δ , so that $\mu_{\delta\mu} = \mu$ and $\delta_{\mu\delta} = \delta$ (cf. Willard [5], section 36). It can be easily verified that $C_{\mu_\delta}(X,Y) = C_\delta(X,Y)$ and $C_{\delta_{\mu}}(X,Y) = C_\mu(X,Y)$. Therefore a uniform structure on Y may be considered either as a diagonal uniformity or a covering uniformity, and the resulting uniform structures on $C(X,Y)$ will generate the same topology, called the uniform topology.

Let ν stand for the fine (covering) uniformity on Y . Whenever Y is paracompact, ν has as a base the family of all open covers of Y . The topology on $C_\nu(X,Y)$ will be called the fine uniform topology.

If μ is any compatible uniformity on Y , then the relationships between the various topologies discussed above are given by

$$C_p(X,Y) \leq C_k(X,Y) \leq C_\mu(X,Y) \leq C_\nu(X,Y), \quad (1.3)$$

where the inequality means the space on the right is finer than that on the left.

Each compatible bounded metric ρ on Y induces the supremum metric $\hat{\rho}$ on $C(X,Y)$, defined by $\hat{\rho}(f,g) = \sup\{\rho(f(x),g(x)) : x \in X\}$. The resulting topological space will be denoted by $C_\rho(X,Y)$. A base for $C_\rho(X,Y)$ consists of the metric balls $\{B_\rho(f,e) : f \in C(X,Y) \text{ and } e \in \mathbb{R}^+\}$. If μ_ρ is the uniformity on Y generated by ρ , then $C_{\mu_\rho}(X,Y) = C_\rho(X,Y)$.

For a metrizable space Y , let $M(Y)$ be the family of all compatible bounded metrics on Y . The following fact gives a useful tool for working with the fine uniform topology.

PROPOSITION 1.1. If Y is metrizable, then $C_\nu(X,Y)$ has as a base $\{B_\rho(f,e) : \rho \in M(Y), f \in C(X,Y) \text{ and } e \in \mathbb{R}^+\}$.

PROOF. To see that $B_\rho(f,e)$ is a neighborhood of f in $C_\nu(X,Y)$, define the open cover $U = \{B_\rho(y,e/3) : y \in Y\}$ of Y , and let $g \in U[f]$. Then for each $x \in X$, there is a $y \in Y$ with $(f(x), g(x)) \in B_\rho(y, e/3)$. Therefore each $\rho(f(x),g(x)) < 2e/3$, so that $\hat{\rho}(f,g) \leq 2e/3 < e$. This establishes that $g \in B_\rho(f,e)$, and it follows that $\hat{U}[f] \subset B_\rho(f,e)$.

On the hand, let $U \in \mu$ and $f \in C(X,Y)$. Let $U_1 >^* U_2 >^* \dots$ be a normal sequence of open covers of Y so that U_1 refines U , and let ρ be the metric defined by this sequence (see Willard [5], p. 167, for this construction). It follows from the construction of ρ that there is an $e \in \mathbb{R}^+$ so that $B_\rho(f,e) \subset \hat{U}[f]$.

Let μ be any compatible uniformity on Y . If X is compact then $\hat{\mu}$ on $C(X,Y)$ is the same as the uniformity of uniform convergence on compact sets, which is known to generate the compact-open topology (cf. Willard [5], section 43). The converse is in fact also true for most Y .

PROPOSITION 1.2. If Y contains a nontrivial path, then for any compatible uniformity μ on Y , $C_\mu(X,Y) = C_k(X,Y)$ if and only if X is compact.

PROOF. Let $\phi : I \rightarrow Y$ be a continuous function from the closed unit interval into Y such that $\phi(0) \neq \phi(1)$. Let $y = \phi(0)$, let $z = \phi(1)$, and let f be the constant map from X to y . Define $U = \{Y \setminus \{y\}, Y \setminus \{z\}\}$, which is an open cover of Y . To see that $U \in \mu$, let $D \in \delta_\mu$ with $z \notin D[y]$. Choose a symmetric $E \in \delta_\mu$ with $E \cap E \subset D$. It follows that $\{E[p] : p \in Y\}$ refines U , so that $U \in \mu$.

Suppose that X is not compact. To see that $\hat{U}[f]$ is not a neighborhood of f in $C_k(X,Y)$, let $W = [A_1, V_1] \cap \dots \cap [A_n, V_n]$ be any basic open subset of $C_k(X,Y)$ which contains f , where each $[A_i, V_i] = \{f \in C(X,Y) : f(A_i) \subset V_i\}$ and the A_i are compact and the V_i are open. If $A = A_1 \cup \dots \cup A_n$, then there is some $x \in X \setminus A$. Define $g: A \cup \{x\} \rightarrow Y$ by $g(a) = y$ for $a \in A$ and $g(x) = z$. Since $\phi(I)$ is arcwise connected, then g extends to some $\hat{g} \in C(X,Y)$. Then $\hat{g} \in W \setminus \hat{U}[f]$, which establishes that $C_\mu(X,Y)$ is finer than $C_k(X,Y)$.

It follows from Proposition 1.2 that whenever X is compact, all compatible uniformities on Y generate the same topology on $C(X,Y)$ - the compact-open topology. Also when Y is compact, a compatible uniformity on Y generates a unique topology on $C(X,Y)$ since there is only one compatible uniformity on Y - but in this case the topology on $C(X,Y)$ is not the compact-open topology, unless X is also compact.

PROPOSITION 1.3. Let Y be a metrizable space. If X is pseudocompact, then all compatible uniformities on Y generate the same topology on $C(X,Y)$.

PROOF. Let U be an open cover of \mathbb{R} , and let $f \in C(X,Y)$. For each $x \in X$, let $V_x \in U$ with $f(x) \in V_x$. Then for each such x there exists a $U_x \in U$ with $(D_x \circ D_x)[f(x)] \subset V_x$, where $D_x = \cup\{U^2 : U \in U_x\}$. Since X is pseudocompact and Y is metrizable, then $f(X)$ is compact. So there exist $x_1, \dots, x_n \in X$ such that $f(X) \subset D_{x_1}[f(x_1)] \cup \dots \cup D_{x_n}[f(x_n)]$. Now there exists an $U_0 \in U$ which refines each of U_{x_1}, \dots, U_{x_n} .

In order to show that $\hat{U}_0[f] \subset \hat{U}[f]$, let $g \in \hat{U}_0[f]$ and $x \in X$. Then there exists an i such that $f(x) \in D_{x_1}[f(x_1)]$. Let $U \in U_0$ with $(f(x), g(x)) \in U^2$. There is some $U_i \in U_{x_1}$ such that $U \subset U_i$, so that $(f(x), g(x)) \in U_i^2 \subset D_{x_1}$. Therefore $f(x) \in D_{x_1}[f(x_1)] \subset (D_{x_1} \circ D_{x_1})[f(x_1)] \subset V_{x_1}$ and $g(x) \in (D_{x_1} \circ D_{x_1})[f(x_1)] \subset V_{x_1}$. Since x is arbitrary, $g \in \hat{U}[f]$.

The topology generated in Proposition 1.3 is in general not the compact-open topology.

The "completeness" of a function space can be a useful property for obtaining the existence of certain kinds of functions. If ρ is a complete metric on Y , then $\hat{\rho}$ is a complete metric on $C(X,Y)$. On the other hand, $C_\nu(X,Y)$ may not be metrizable. So complete metrizability of $C_\nu(X,Y)$ is too much to expect in general. But $C_\nu(X,Y)$ does have "completeness" to the following extent.

THEOREM 1.4. If Y is completely metrizable, then $C_\nu(X,Y)$ is a Baire space.

PROOF. Let ρ be a compatible bounded complete metric on Y . Also let $\{W_n : n \in \omega\}$ be a sequence of dense open subsets of $C_\nu(X,Y)$, and let W be a nonempty open subset of $C_\nu(X,Y)$. Choose $d_1 \in M(Y)$, $f_1 \in C(X,Y)$, and $0 < \epsilon_1 < 1/2$ so that $B_{d_1}(f_1, \epsilon_1) \subset W_1 \cap W$. Define $\rho_1 = \max\{\rho, d_1\}$, so that $B_{\rho_1}(f_1, \epsilon_1) \subset B_{d_1}(f_1, \epsilon_1)$. Continue by induction so that at the $n+1$ step, choose $d_{n+1} \in M(Y)$, $f_{n+1} \in C(X,Y)$, and $0 < \epsilon_{n+1} < 1/2^{n+1}$ such that $B_{d_{n+1}}(f_{n+1}, \epsilon_{n+1}) \subset W_{n+1} \cap B_{\rho_n}(f_n, \epsilon_n/2)$; and define $\rho_{n+1} = \max\{\rho_n, d_{n+1}\}$. Now $\{f_n : n \in \omega\}$ is a Cauchy sequence in $C_\rho(X,Y)$, and therefore converges to some $f \in C_\rho(X,Y)$. Also for each $i \in \omega$, $\{f_n : n \in \omega\}$ will converge to f in $C_{\rho_i}(X,Y)$, so that f is in the closure of $B_{\rho_i}(f_i, \epsilon_i/2)$ in $C_{\rho_i}(X,Y)$. Therefore $f \in B_{\rho_i}(f_i, \epsilon_i) \subset B_{d_i}(f_i, \epsilon_i) \subset W_i \cap W$ for every $i \in \omega$.

The conclusion of Theorem 1.4 cannot be strengthened to Cech-completeness, which can be seen from Theorem 4.1. Also the hypothesis cannot be changed to Y being compact. For example, if X is the closed unit interval I with the usual topology, and $Y = I^2$

with the order topology with respect to lexicographic ordering, then some functions in $C_{\vee}(X, Y)$ have open neighborhoods which can be written as countable unions of nowhere dense sets.

2. FINE TOPOLOGIES.

Throughout this section, (Y, ρ) will be a metric space. In this case there is a natural way to generate a topology on $C(X, Y)$ which may be even finer than the fine uniformity topology.

For each $f \in C(X, Y)$ and $\phi \in C^+(X)$, define $B_{\rho}^+(f, \phi) = \{g \in C(X, Y) : \text{for every } x \in X, \rho(f(x), g(x)) < \phi(x)\}$. Since for each $\phi, \psi \in C^+(X)$, $\max\{\phi, \psi\} \in C^+(X)$, then $\{B_{\rho}^+(f, \phi) : f \in C(X, Y) \text{ and } \phi \in C^+(X)\}$ is a base for a topology on $C(X, Y)$. This topology is called the fine topology with respect to ρ (Munkres [3], p.285), and will be denoted by $C_{f_{\rho}}(X, Y)$. Certainly $C_{f_{\rho}}(X, Y)$ is finer than $C_{\rho}(X, Y)$, and is in general strictly finer. However, for pseudocompact X , they are the same.

PROPOSITION 2.1. If X is pseudocompact, then $C_{f_{\rho}}(X, Y) = C_{\rho}(X, Y)$.

PROOF. Let $f \in C_{f_{\rho}}(X, Y)$, and let $\phi \in C^+(X)$. Since $1/\phi \in C^+(X)$ and since X is pseudocompact, then $1/\phi$ is bounded. So there is a number M with $1/\phi(x) < M$ for all $x \in X$. Therefore $B_{\rho}(f, 1/M) \subset B_{\rho}^+(f, \phi)$.

From Propositions 1.3 and 2.1 it follows that if X is pseudocompact then $C_{f_{\rho}}(X, Y) = C_{\vee}(X, Y)$. But even if X is not pseudocompact, $C_{f_{\rho}}(X, Y)$ can still be related to $C(X, Y)$ as follows.

PROPOSITION 2.2. If X is paracompact, then $C_{\vee}(X, Y) \subset C_{f_{\rho}}(X, Y)$.

PROOF. Let $d \in M(Y)$, let $f \in C(X, Y)$, and let $\epsilon \in \mathbb{R}^+$. For each $x \in X$, there exists a $\psi(x) \in \mathbb{R}^+$ such that $B_{\rho}(f(x), \psi(x)) \subset B_d(f(x), \epsilon/2)$. Let $U = \{U_{\alpha} : \alpha \in A\}$ be a star-refinement of $\{f^{-1}(B_{\rho}(f(x), \psi(x)/2)) : x \in X\}$. Let $\{\phi_{\alpha} : \alpha \in A\}$ be a partition of unity subordinated to U . For each $\alpha \in A$, let n_{α} be the cardinality of $\{\beta \in A : U_{\alpha} \cap U_{\beta} \neq \emptyset\}$, let $x_{\alpha} \in X$ be such that $U_{\alpha} \subset f^{-1}(B_{\rho}(f(x_{\alpha}), \psi(x_{\alpha})))$, and let $m_{\alpha} = \min\{\psi(x_{\beta})/2 : U_{\alpha} \cap U_{\beta} \neq \emptyset\}$. Then define $\phi = \sum\{(m_{\alpha}/n_{\alpha})\phi_{\alpha} : \alpha \in A\}$, which is a member of $C^+(X)$.

To establish that $B_{\rho}^+(f, \phi) \subset B_d(f, \epsilon)$, let $g \in B_{\rho}^+(f, \phi)$ and let $x \in X$. Also let $U_{\alpha_1}, \dots, U_{\alpha_k}$ be the members of U containing x . Take i to be such that $m_{\alpha_1}\phi_{\alpha_1}(x) = \min\{m_{\alpha_1}\phi_{\alpha_1}(x), \dots, m_{\alpha_k}\phi_{\alpha_k}(x)\}$. Then $\phi(x) \leq m_{\alpha_1}\phi_{\alpha_1}(x)/(1/n_{\alpha_1} + \dots + 1/n_{\alpha_k}) \leq m_{\alpha_1}\phi_{\alpha_1}(x) \leq \psi(x_{\alpha_1})/2$. It follows that $\rho(g(x), f(x)) < \psi(x_{\alpha_1})/2$. Since $f(x) \in B_{\rho}(f(x_{\alpha_1}), \psi(x_{\alpha_1})/2)$, then $g(x) \in B_{\rho}(f(x_{\alpha_1}), \psi(x_{\alpha_1})) \subset B_d(f(x_{\alpha_1}), \epsilon/2)$. Also $f(x) \in B_d(f(x_{\alpha_1}), \epsilon/2)$, so that $g(x) \in B_d(f(x), \epsilon)$.

The inequality in Proposition 2.2 is in general not an equality as indicated in the comment after Corollary 3.5.

Whenever ρ is complete then $C_{f_{\rho}}(X, Y)$ is a Baire space ([4], p.297). Instead of giving a proof of this here, a proof will be given in the next section that $C_{f_{\rho}}(X)$ is pseudo-complete, which is a property stronger than being a Baire space.

3. REAL-VALUED FUNCTIONS. For the rest of the paper, ρ will denote the usual metric on \mathbb{R} bounded by 1; that is, $\rho(s, t) = \min\{1, |s-t|\}$.

LEMMA 3.1. Let $f, g \in C(X)$ and let $\phi, \psi \in C^+(X)$. Then the closure of $B_{\rho}^+(f, \phi)$ in $C_{f_{\rho}}(X)$ is contained in $B_{\rho}^+(g, \psi)$ if and only if for each $x \in X$ the closure of $B_{\rho}(f(x), \phi(x))$ in \mathbb{R} is contained in $B_{\rho}(g(x), \psi(x))$.

PROOF. For the sufficiency, let $h \in B_\rho^+(f, \phi)$ and let $x \in X$. Suppose $h(x)$ were contained in the complement of the closure of $B_\rho(f(x), \phi(x))$; call this set V . Then the set of functions taking x into V would be a neighborhood of h in $C_{f_\rho}(X)$ which misses $B_\rho^+(f, \phi)$. This contradiction shows that $h(x) \in \text{cl}(B_\rho(f(x), \phi(x))) \subset B_\rho(g(x), \phi(x))$.

For necessity, let $x_0 \in X$ and let $t \in \text{cl}(B_\rho(f(x_0), \phi(x_0)))$ in \mathbb{R} . Define $h(x) = f(x) + (t - f(x_0)) \cdot \phi(x) / \phi(x_0)$ for all $x \in X$. This defines an $h \in C(X)$ such that $h(x_0) = t$. Also $|h(x) - f(x)| \leq \phi(x)$ for each x .

To see that $h \in \text{cl}(B_\rho^+(f, \phi))$ in $C_{f_\rho}(X)$, let $B_\rho^+(h, \xi)$ be a basic neighborhood of h . Define $k(x) = h(x) + \text{sign}(f(x) - h(x)) \cdot \min\{\xi(x)/2, |h(x) - f(x)|\}$, which is an element of $C(X)$. Now $|k(x) - h(x)| = \min\{\xi(x)/2, |h(x) - f(x)|\} \leq \xi(x)/2 < \xi(x)$. Also if $h(x) > f(x)$, then $|k(x) - f(x)| = |h(x) - f(x) - \min\{\xi(x)/2, h(x) - f(x)\}|$. If this is positive, then it is equal to $\xi(x)/2$, and $\xi(x)/2 \leq |h(x) - f(x)| < \phi(x)$. The same argument shows that if $h(x) < f(x)$, then also $|k(x) - f(x)| < \phi(x)$. Therefore $k \in B_\rho^+(f, \phi)$, so that $B_\rho^+(h, \xi) \cap B_\rho^+(f, \phi) \neq \emptyset$. From this it follows that $h \in \text{cl}(B_\rho^+(f, \phi)) \subset B_\rho^+(g, \psi)$. But then $|t - g(x_0)| = |h(x_0) - g(x_0)| < \psi(x_0)$, so that $t \in B_\rho(g(x_0), \psi(x_0))$.

The proof of Lemma 3.1 depends on the metric (and algebraic) structure of the range space. In fact the lemma is not true in general. For example, let $X = I$ and $Y = \{(s, \sin(2\pi/s)) \in \mathbb{R}^2 : 0 < s \leq 2\} \cup \{(0, 0)\}$ with metric d on Y defined by $d((s_1, t_1), (s_2, t_2)) = \max\{|s_1 - s_2|, |t_1 - t_2|\}$. Also let $y = (2, 0)$, let $z = (0, 0)$, let f be the constant map taking X to y , and let g be the constant map taking X to z . Then $B_d(f, 2) = C(X, Y) \setminus \{g\}$. Since g is isolated, $B_d(f, 2)$ is closed, so that $\text{cl}(B_d(f, 2)) \subset B_d(f, 2)$. On the other hand, for any $x \in X$, $z \in \text{cl}(B_d(f(x), 2)) \setminus B_d(f(x), 2)$ in Y .

THEOREM 3.2. The space $C_{f_\rho}(X)$ is pseudo-complete.

PROOF. For each $n \in \omega$, let $C_n^+(X) = \{\phi \in C^+(X) : \text{for all } x \in X, \phi(x) < 1/2^n\}$, and define $B_n = \{B_\rho^+(f, \phi) : f \in C(X) \text{ and } \phi \in C_n^+(X)\}$. Each B_n is a base for $C_{f_\rho}(X)$. It remains to show that if $B_n \in B_n$ for each n with $\text{cl}(B_n) \subset B_n$ then $\cap\{B_n : n \in \omega\} \neq \emptyset$. If each $B_n = B_\rho^+(f_n, \phi_n)$, then by Lemma 3.1, for each $n \in \omega$ and each $x \in X$, $\text{cl}(B_\rho(f_{n+1}(x), \phi_{n+1}(x))) \subset B_\rho(f_n(x), \phi_n(x))$. Since each $\phi_n(x) < 1/2^n$, then $\cap\{B_\rho(f_n(x), \phi_n(x)) : n \in \omega\} = \{f(x)\}$ for some $f(x) \in \mathbb{R}$. This defines the function f , which is the uniform limit of $\{f_n : n \in \omega\}$, and is hence continuous. Clearly $f \in \cap\{B_n : n \in \omega\}$, as desired.

The algebraic structure on \mathbb{R} induces an algebraic structure on $C(X)$. This structure interacts well with some topologies on $C(X)$. For example, $C_k(X)$ and $C_p(X)$ are always locally convex linear topological spaces. On the other hand, $C_\rho(X)$ is only a topological group under addition while the scalar multiplication operation is not continuous for non-compact X . The space $C_{f_\rho}(X)$ behaves much like $C_\rho(X)$ in this regard. It is straightforward to show that $C_{f_\rho}(X)$ is a topological group under addition. As a result, $C_{f_\rho}(X)$ is homogeneous; and for many arguments it suffices to consider only basic neighborhoods of the zero function, f_0 .

The next result establishes when f_0 has a countable base. It is stated for (\mathbb{R}, ρ) , but it is also true for any metric space containing a nontrivial path.

PROPOSITION 3.3. If X is normal and f_0 has a countable base in $C_{f_\rho}(X)$, then X is countably compact.

PROOF. Suppose X is not countably compact. Then X contains a countable closed discrete set, $\{x_n : n \in \omega\}$. Let $\{\phi_n : n \in \omega\}$ be any sequence in $C^+(X)$. The goal is to show that $\{B_\rho^+(f_0, \phi_n) : n \in \omega\}$ cannot be a base at f_0 . For each n , let $e_n \in \mathbb{R}^+$ be such that interval $[0, e_n]$ is contained in $B_\rho(0, \phi_n(x_n)/2)$. Since X is normal, there exists a $\phi \in C^+(X)$ such that $\phi(x_n) = e_n$ for every n .

It remains to show that $B_\rho^+(f_0, \phi_n) \not\subset B_\rho^+(f, \phi)$ for each n . Fix $n \in \omega$, and let U be a neighborhood of x_n such that $\phi_n(U) \subset (\phi_n(x_n)/2, \infty)$. Because X is a Tychonoff space, there exists an $f \in C(X)$ such that $f(x_n) = e_n$, $f(X \setminus U) = \{0\}$, and $f(U) \subset [0, e_n]$. Clearly $f \notin B_\rho^+(f_0, \phi)$ since $f(x_n) = e_n = \phi(x_n)$. To see that $f \notin B_\rho^+(f_0, \phi_n)$, let $x \in X$. If $x \notin U$, then $\rho(f(x), f_0(x)) = 0$. If $x \in U$, then $f(x) \in [0, e_n] \subset B_\rho(0, \phi_n(x_n)/2)$. Also $\phi_n(x) > \phi_n(x_n)/2$, so that $\rho(f(x), f_0(x)) < \phi_n(x)$.

Therefore for a normal space X , $C_{f_\rho}^+(X)$ is first countable if and only if it is already equal to the metrizable space $C_\rho(X)$.

The situation is little different for $C_\nu(X)$. To begin with, $C_\nu(X)$ is in general not homogeneous, as the next proposition shows.

PROPOSITION 3.4. An element of $C_\nu(X)$ has a countable base if and only if it is a bounded function.

PROOF. First suppose f is an unbounded function in $C(X)$. Without loss of generality, suppose there is a sequence $\{x_n : n \in \omega\}$ in X such that $f(x_n) = n$ for each $n \in \omega$. Let $\{U_n[f] : n \in \omega\}$ be any sequence of basic neighborhoods of f in $C_\nu(X)$.

For each $n \in \omega$, let $V_n \in U_n$ with $n \in V_n$, and let I_n be a closed interval containing n in its interior and contained in $V_n \cap (n-1/2, n+1/2)$. Also let t_n be a point of the interior of I_n different than n , and W_n be an open subset of the interior of I_n which contains n but not t_n . Then let $W = \mathbb{R} \setminus \omega$, and define $U = \{W\} \cup \{W_n : n \in \omega\}$.

To see that each $\hat{U}_n[f]$ is not contained in $\hat{U}[f]$, let $\phi_n : I_n \rightarrow I_n$ be a homeomorphism which fixes the endpoints of I_n and moves n to t_n . Then define $\phi \in C(\mathbb{R})$ by $\phi(s) = \phi_n(s)$ if $s \in I_n$ and $\phi(s) = s$ otherwise. It follows that $\phi \circ f \in \hat{U}_n[f] \setminus \hat{U}[f]$. Therefore $\{\hat{U}_n[f] : n \in \omega\}$ cannot be a base at f .

For the converse, suppose that f is a bounded function in $C(X)$. The goal will be to show that $\{B_\rho(f, 1/n) : n \in \omega\}$ is a base at f in $C_\nu(X)$. Let $d \in M(\mathbb{R})$ and let $\epsilon > 0$. Also let $M \in \omega$ such that $f(X)$ is contained in the interval $[-M, M]$. For each $t \in [-M, M]$, there exists an $e_t \in \mathbb{R}^+$ such that $B_\rho(t, e_t) \subset B_d(t, \epsilon/3)$. There exist $t_1, \dots, t_m \in [-M, M]$ such that $[-M, M] \subset B_\rho(t_1, e_{t_1}/2) \cap \dots \cap B_\rho(t_m, e_{t_m}/2)$. Take $n \in \omega$ with $1/n \leq \min\{e_{t_1}/2, \dots, e_{t_m}/2\}$.

To see that $B_\rho(f, 1/n) \subset B_d(f, \epsilon)$, let $g \in B_\rho(f, 1/n)$ and let $x \in X$. There is some k with $f(x) \in B_\rho(t_k, e_{t_k}/2)$. Now $\rho(g(x), f(x)) < 1/n \leq e_{t_k}/2$ and $\rho(f(x), t_k) < e_{t_k}/2$, so that $\rho(g(x), t_k) < e_{t_k}$. Therefore $g(x) \in B_\rho(t_k, e_{t_k}) \subset B_d(t_k, \epsilon/3)$, and similarly $f(x) \in B_d(t_k, \epsilon/3)$. Hence $d(g(x), f(x)) < 2\epsilon/3$, so that $d(g, f) \leq 2\epsilon/3 < \epsilon$.

COROLLARY 3.5. If $C_\nu(X)$ is first countable, then X is pseudocompact.

It follows from Proposition 3.4 that whenever X is not pseudocompact then $C_\nu(X)$ is not homogeneous, and is thus not a topological group under addition. Therefore $C_\nu(X)$ and $C_{f_\rho}(X)$ are different whenever X is not pseudocompact.

4. COUNTABILITY PROPERTIES

The spaces $C_\nu(X)$ and $C_{f_\rho}(X)$ have a property which is useful for studying countability properties. That property is submetrizability; i.e., these topologies contain weaker metrizable topologies. Certain results follow immediately. For example, singleton sets in $C_\nu(X)$ and $C_{f_\rho}(X)$ are G_δ -sets. Also the concepts of compactness, countable compactness, and sequential compactness are equivalent for subsets of these spaces.

There is a concept which is weaker than first countability that will be useful to consider. A space is of point countable type if every point is contained in a compact set which has a countable base. Every Cech-complete space has this property. Also a space which is of point countable type and in which singleton sets are G_δ -sets is first countable. As a result, a number of properties are equivalent for the fine and fine uniform topologies. The proof of the following theorem then follows from Proposition 1.3 and Corollary 3.5.

THEOREM 4.1. If $C(X)$ has the fine uniform topology (or the fine topology for normal X), then the following are equivalent.

- (a) $C(X)$ is first countable.
- (b) $C(X)$ is of point countable type.
- (c) $C(X)$ is Cech-complete
- (d) $C(X)$ is metrizable.
- (e) $C(X)$ is completely metrizable.
- (f) $C(X) = C_\rho(X)$.
- (g) All compatible uniformities on \mathbb{R} induce the same topology on $C(X)$.
- (h) X is pseudocompact.

Theorem 4.1 is also true with \mathbb{R} replaced by any complete metric space which contains a closed ray; i.e., a closed copy of the interval $[0, \infty)$.

THEOREM 4.2. If $C(X)$ has the fine uniform topology (or the fine topology), then the following are equivalent.

- (a) $C(X)$ is separable.
- (b) $C(X)$ has the countable chain condition.
- (c) $C(X)$ is Lindelof.
- (d) $C(X)$ has a countable network.
- (e) $C(X)$ is second countable.
- (f) $C(X)$ is separable and completely metrizable.
- (g) $C_\rho(X)$ is separable.
- (h) X is compact and metrizable.

PROOF. Since $C_\rho(X) \leq C(X)$, each of (a) through (f) implies (g). That (g) implies (h) is well-known. Finally, if (h) is true, then $C(X) = C_\rho(X)$.

These same arguments can be extended to generalize Theorem 4.2. In particular, this theorem will be true if \mathbb{R} is replaced by a separable complete metric space which contains a nontrivial path.

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