

A SIMULTANEOUS SOLUTION TO TWO PROBLEMS ON DERIVATIVES

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ABSTRACT. Let A be a nonvoid countable subset of the unit interval $[0,1]$ and let B be an F_σ -subset of $[0,1]$ disjoint from A . Then there exists a derivative f on $[0,1]$ such that $0 \leq f \leq 1$, $f = 0$ on A , $f > 0$ on B , and such that the extended real valued function $1/f$ is also a derivative.

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In this note, we construct a derivative f such that $1/f$ is also a derivative, and f and $1/f$ have some curious properties mentioned in [1] and [2]. (By an F_σ -set in the real line, we mean the union of countably many closed subsets of \mathbb{R} .) We prove

THEOREM 1. Let A be a nonvoid countable subset of $[0,1]$ and let B be an F_σ -subset of $[0,1]$ disjoint from A . Then there exists a measurable function f on $[0,1]$ such that $f = 0$ on A , $f > 0$ on B , $0 \leq f \leq 1$ on $[0,1]$ and

- (1) f is everywhere the derivative of its primitive,
- (2) $1/f$ is Lebesgue summable on $[0,1]$,
- (3) $1/f$ is everywhere the derivative of its primitive.

Here we let $\infty = 1/0$.

When $m(B) = 1$ and A is dense, we will obtain a simple example of a derivative that vanishes on a dense set of measure 0.

From [2] we infer that there exists a derivative f vanishing on A and positive on B . From [1] we infer that there exists a derivative g infinite on A and finite on B . However, Theorem 1 provides a simultaneous solution to both of these problems. To prove Theorem 1 we will employ some of the methods used in [3].

Finally, we use these methods to construct a concrete example of functions g_1 and g_2 that have finite or infinite derivatives at each point, such that the Dini derivatives of their difference, $g_1 - g_2$, satisfy certain pathological properties.

In all that follows, let $(n(i))_{i=1}^{\infty}$ denote the sequence of integers 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1,

Proof of Theorem 1. Let $(a_i)_{i=1}^{\infty}$ be a sequence of points in A such that each point of A occurs at least once in the sequence. (Here we do not exclude the possibility that A is a finite set.) We assume, without loss of generality, that B is nonvoid.

Let $B_1 \subset B_2 \subset B_3 \subset \dots \subset B_i \dots$ be an expanding sequence of closed sets such that $B = \bigcup_{i=1}^{\infty} B_i$. (Here we do not exclude the possibility that B is a closed set.) Let u_i denote the distance from the point $a_{n(i)}$ to the set B_i . As in [3], we put $\emptyset(x) = (1 + |x|)^{-\frac{1}{2}}$.

For each index j , put

$$g_j(x) = 1 + \sum_{i=1}^j \emptyset(2^i u_i^{-1}(x - a_{n(i)})),$$

$$g(x) = 1 + \sum_{i=1}^{\infty} \emptyset(2^i u_i^{-1}(x - a_{n(i)})),$$

$$f_j(x) = 1/g_j(x), \quad f(x) = 1/g(x).$$

Here we let $0 = 1/\infty$. Then $g(a) = \infty$ for $a \in A$, because there are infinitely many indices i for which $a = a_{n(i)}$. On the other hand, $g(b) < \infty$ for $b \in B$; note that if $b \in B_k$, then

$$\emptyset(2^k u_k^{-1}(b - a_{n(k)})) \leq \emptyset(2^k) < 2^{-\frac{1}{2}k},$$

$$\sum_{i=k}^{\infty} \emptyset(2^i u_i^{-1}(b - a_{n(i)})) \leq \sum_{i=k}^{\infty} 2^{-\frac{1}{2}i} < \infty.$$

We infer from Lemma 4 of [3], that g is Lebesgue summable on $[0,1]$. Note also that

$$g(x) - g_j(x) = g(x)g_j(x)(f_j(x) - f(x)) > 0,$$

and since $g > 1$, $g_j > 1$, it follows that $g - g_j > f_j - f > 0$.

Now choose any x with $g(x) < \infty$. By Lemma 4 of [3], we have

$$\lim_{h \rightarrow 0} h^{-1} \int_x^{x+h} g(t) dt = g(x).$$

Take any $\epsilon > 0$. Select an index j so large that $f_j(x) - f(x) < g(x) - g_j(x) < \epsilon$. Since f_j and g_j are continuous, when $|h|$ is small enough we have

$$|h^{-1} \int_x^{x+h} g(t) dt - g(x)| < \epsilon,$$

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For such j and h we obtain

$$\begin{aligned} h^{-1} \int_x^{x+h} (g(t) - g_j(t)) dt &\leq g(x) - g_j(x) + |h^{-1} \int_x^{x+h} g(t) dt - g(x)| \\ &\quad + |h^{-1} \int_x^{x+h} g_j(t) dt - g_j(x)| < 3\epsilon. \end{aligned}$$

From $0 < f_j - f < g - g_j$ we obtain

$$\begin{aligned} |h^{-1} \int_x^{x+h} f(t) dt - f(x)| &\leq |h^{-1} \int_x^{x+h} f_j(t) dt - f_j(x)| + f_j(x) - f(x) \\ &\quad + h^{-1} \int_x^{x+h} (f_j(t) - f(t)) dt \\ &\leq 2\epsilon + h^{-1} \int_x^{x+h} (f_j(t) - f(t)) dt \\ &\leq 2\epsilon + h^{-1} \int_x^{x+h} (g(t) - g_j(t)) dt < 5\epsilon. \end{aligned}$$

It follows that $\lim_{h \rightarrow 0} h^{-1} \int_x^{x+h} f(t) dt = f(x)$.

Choose any x with $g(x) = \infty$. Take any $N > 0$. Select j so large that $g_j(x) > N$. Since g_j is continuous, there is a $d > 0$ such that $|t-x| < d$ implies $g_j(t) > N$. For such t , $g(t) > g_j(t) > N$ and $f(t) < f_j(t) < N^{-1}$. It follows that for $|h| < d$,

$$h^{-1} \int_x^{x+h} g(t) dt > N, \quad 0 < h^{-1} \int_x^{x+h} f(t) dt < N^{-1}.$$

Finally,

$$\lim_{h \rightarrow 0} h^{-1} \int_x^{x+h} g(t) dt = \infty = g(x).$$

$$\lim_{h \rightarrow 0} h^{-1} \int_x^{x+h} f(t) dt = 0 = f(x).$$

This completes the proof.

When $m(B) = 1$, we do not know if our argument can be modified to make $f = 0$ on $[0,1] \setminus B$ as in [2]. Perhaps this requires an approach altogether different from ours.

We say that x is a knot point of the function F if its Dini derivatives satisfy

$$D^+F(x) = D^-F(x) = \infty \quad \text{and} \quad D_+F(x) = D_-F(x) = -\infty.$$

We conclude by presenting a simple and direct example of functions g_1 and g_2 having derivatives (finite or infinite) at every point such that $g_1 - g_2$ has knot points in every interval. (Consult [4] for analogous examples.)

Let $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ be countable dense subsets of $(0,1)$ that are disjoint. Let $Z(c,d,x) = \int_0^x \emptyset(c(t-d)) dt$ for $c > 0$, $d > 0$, $x > 0$. We integrate to obtain

$$Z(c,d,x) = \begin{cases} 2c^{-1} [(1+cd)^{\frac{1}{2}} - (1+cd-cx)^{\frac{1}{2}}] & \text{if } x \leq d, \\ 2c^{-1} [(1+cd)^{\frac{1}{2}} + (1+cx-cd)^{\frac{1}{2}} - 2] & \text{if } x > d. \end{cases}$$

Let u_i denote the distance from $a_{n(i)}$ to the set $\{b_1, \dots, b_i\}$, and let v_i denote the distance from $b_{n(i)}$ to the set $\{a_1, \dots, a_i\}$. Put

$$g_1(x) = \sum_{i=1}^\infty Z(2^i u_i^{-1}, a_{n(i)}, x), \quad g_2(x) = \sum_{i=1}^\infty Z(2^i v_i^{-1}, b_{n(i)}, x)$$

for $0 < x < 1$. By the argument in the proof of Theorem 1 we prove that g_1 and g_2 are absolutely continuous functions on $(0,1)$ with $g_1' = \infty$ on A , $g_2' = \infty$ on B , g_1' finite on B , and g_2' finite on A . Put $g = g_1 - g_2$. Then g is absolutely continuous on $(0,1)$, $g' = \infty$ on A and $g' = -\infty$ on B . Each of the sets

$E_1 = \{x: D^+g(x) = \infty\}$, $E_2 = \{x: D^-g(x) = \infty\}$, $E_3 = \{x: D_+g(x) = -\infty\}$ and $E_4 = \{x: D_-g(x) = -\infty\}$ is a dense G_δ -subset of $(0,1)$, i.e., is the intersection of countably many open dense subsets of $(0,1)$. It follows that $E_1 \cap E_2 \cap E_3 \cap E_4$ is also a dense G_δ -subset of $(0,1)$. But each point in this intersection is a knot point of g , even though g_1 and g_2 have derivatives (finite or infinite) everywhere by the proof of Theorem 1.

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