

**TAUBERIAN THEOREM FOR THE DISTRIBUTIONAL
 STIELTJES TRANSFORMATION**

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ABSTRACT. In this paper we use the notion of L-quasiasymptotic at infinity of distributions to obtain a final value Tauberian theorem for the distributional Stieltjes transformation.

KEY WORDS AND PHRASES. Slowly varying function, the quasiasymptotic behaviour at infinity, distributional Stieltjes transform.

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1. NOTIONS AND NOTATION.

Throughout this paper, L will denote real valued, positive and measurable function on $[A, \infty)$ such that

$$\lim_{t \rightarrow \infty} \frac{L(t\lambda)}{L(t)} = 1$$

for each $\lambda > 0$. Function L which is regularly varying with index of regular variation $a = 0$, is called slowly varying and the class of such functions is introduced and investigated by J. Karamata.

The quasiasymptotic behaviour at infinity of tempered distributions with supports in $[0, \infty)$ (denoted by S_+^1) was defined by Zavijalov (see for instance [1]).

Definition 1. A distribution $f \in S_+^1$ has the L-quasiasymptotic at infinity of the power $a \in \mathbb{R}$ and with the limit $g \in S_+^1$, $g \neq 0$, if for every $\phi \in S$ (S is the space of rapidly decreasing functions)

$$\lim_{k \rightarrow \infty} \langle \frac{f(kt)}{k^a L(k)}, \phi(t) \rangle = \langle g(t), \phi(t) \rangle. \quad \Delta$$

From the properties of homogeneous distributions it follows ([1]) that if this limit exists then g is a homogeneous distribution of degree a . Namely, $g(t) = C f_{a+1}(t)$ for some $C \neq 0$, where

$$f_{a+1}(t) = \begin{cases} \frac{H(t)t^a}{\Gamma(a+1)} : a > -1 \\ D^n f_{a+n+1}(t) : a \leq -1, a+n > -1. \end{cases}$$

As usual, H is the characteristic function of the interval $(0, \infty)$ and D stands for the distributional derivative.

We use the definition of the distributional Stieltjes transform given in [2], [3] in a little different notation ([4]).

Space $J'(\rho)$, $\rho \in \mathbb{R} \setminus (-N_0)$ ($N_0 = \mathbb{N} \cup \{0\}$) is the space of distributions with supports in $[0, \infty)$ such that $f \in J'(\rho)$ iff there exist $k \in N_0$ and a locally integrable function F with the support in $[0, \infty)$ such that

$$(a) \quad f = D^k F; \quad (b) \quad \int_0^\infty |F(t)| (t+\beta)^{-\rho-k} dt < \infty \quad \text{for } \beta > 0 \quad (1.1)$$

(D is the distributional derivative).

If instead of (b) we suppose that there exist $C = C(F)$ and $\varepsilon = \varepsilon(F) > 0$ such that

$$(c) \quad |F(x)| \leq C(1+x)^{\rho+k-1-\varepsilon} \quad \text{if } x > 0,$$

the corresponding space is denoted by $I'(\rho)$.

Obviously $I'(\rho) \subset J'(\rho)$, $\rho \in \mathbb{R} \setminus (-N_0)$.

It was proved in [4] that:

If $f \in J'(\rho)$, $\rho+k > 0$ (k is from (a)), then $f \in I'(\tilde{\rho})$ for $\tilde{\rho} > \rho$ and $\tilde{\rho} \in \mathbb{R} \setminus (-N_0)$. (1.2)

If $f \in J'(\rho)$, $\rho+k > 0$, then $f \in I'(\tilde{\rho})$ for $\tilde{\rho} > -k$ (k is from (a)) and $\tilde{\rho} \in \mathbb{R} \setminus (-N_0)$. (1.2)

The Stieltjes transform S_ρ of index ρ , $\rho \in \mathbb{R} \setminus (-N_0)$ of a distribution $f \in J'(\rho)$ with the properties given in (1.1) is a complex-valued function given by

$$(S_\rho \{f\})(s) = (\rho)_k \int_0^\infty \frac{F(t) dt}{(t+s)^{\rho+k}}, \quad s \in \mathbb{C} \setminus (-\infty, 0] \quad (1.3)$$

where $(\rho) = \rho(\rho+1) \dots (\rho+k-1)$, $k > 0$ and $(\rho)_0 = 1$.

It is proved in [3] that $(S_\rho \{f\})(s)$ is a holomorphic function of the complex variable s in the domain $\mathbb{C} \setminus (-\infty, 0]$ provided that $f \in J'(\rho)$.

Observe that $f \in J'(\rho)$ implies that $f \in J'(\rho+n)$, $n \in \mathbb{N}$.

The following equality holds:

$$(\rho)_n (S_{\rho+n} \{f\})(s) = (S_\rho \{D^n f\})(s), \quad f \in J'(\rho) \quad \text{and } n \in \mathbb{N}.$$

We shall need the following theorem ([5], p. 339, Macaev and Palant)

THEOREM A. Let us suppose that for some $m > 0$ and $x \rightarrow \infty$

$$\int_0^\infty \frac{d\phi(\lambda)}{(\lambda+x)^{m+1}} \quad \text{and} \quad \int_0^\infty \frac{d\psi(\lambda)}{(\lambda+x)^{m+1}}$$

and the following conditions are satisfied:

- 1) Functions ϕ and ψ are defined for $x > 0$ and are non-decreasing;
- 2) $\lim_{x \rightarrow \infty} (x) = \infty$;
- 3) For any $C > 1$ there are constants γ and N , $0 < \gamma < m$, $N > 0$, such that for any $x < y < N$ is

$$\frac{\phi(x)}{\phi(y)} \leq C \left(\frac{x}{y}\right)^\gamma.$$

Then, for $\lambda \rightarrow \infty$, $\phi(\lambda) \sim \psi(\lambda)$. (This means $\left| \frac{\phi(\lambda)}{\psi(\lambda)} - 1 \right| < \epsilon$ if $\lambda < \lambda_0(\epsilon)$.) Δ

2. TAUBERIAN THEOREM.

THEOREM. Let us suppose that $s > 1$, $\rho+k-s-1 > 0$, $f \in I'(\rho)$ and F (from (1.1)(a)) is non-decreasing function. Moreover, let

$$(S_{\rho}\{f\})(x) \sim \frac{1}{x^s L(x)}, \quad x \rightarrow \infty$$

where L is slowly varying function in some interval $[A, \infty)$, such that $x^{\rho+k-s-1} L(x)$ is non-decreasing function.

Then f has the L -quasiasymptotic of the power $\rho-1-s$ and with the limit

$$\frac{(\rho-s)_k}{(\rho)_k} x^{\rho-1-s}.$$

PROOF. Let us put

$$\phi(x) = \begin{cases} x^{\rho+k-s-1} L(x) & ; \quad x > A \\ 0 & ; \quad x \leq A \end{cases} \tag{2.1}$$

Then ϕ has the L -quasiasymptotic of the power $\rho+k-s-1$ ([1] Theorem 1) and with the limit $x^{\rho+k-s-1}$. By [6] it is

$$\int_0^{\infty} \frac{d\phi(t)}{(x+t)^{\rho+k-1}} = (\rho+k-1) \int_0^{\infty} \frac{\phi(t)dt}{(x+t)^{\rho+k}} \sim \frac{(\rho+k-1)}{x^s L(x)}, \quad x \rightarrow \infty. \tag{2.2}$$

Now we show that the conditions of Theorem A hold for ϕ and F . In fact we have only to show that for some $0 < \gamma < \rho+k-2$ and every $C > 1$ there exists $N > 0$ such that

$$\frac{\phi(\lambda y)}{\phi(y)} < C \lambda^{\gamma} \quad \text{for } \lambda > 1 \quad \text{and } y > N. \tag{2.3}$$

Let us put $\gamma = \rho+k-s-1+\epsilon$ where we choose $\epsilon > 0$ such that $\gamma > 0$ and $\epsilon < s-1$. After substituting (2.1) in (2.3) we obtain

$$L(\lambda y) \leq C \lambda^{\epsilon} L(y)$$

and this inequality is true if $\lambda > 1$ and $y > N$ where N depends on C (see[6]).

From the assumption that $f \in I'(\rho)$ and (2.2) we have

$$(S_{\rho}\{f\})(x) = (\rho)_k \int_0^{\infty} \frac{dF(t)}{(x+t)^{\rho+k}} = (\rho)_{k-1} \int_0^{\infty} \frac{dF(t)}{(x+t)^{\rho+k-1}} \sim \frac{1}{x^s L(x)}, \quad x \rightarrow \infty.$$

It implies

$$(\rho)_k \int_0^{\infty} \frac{dF(t)}{(x+t)^{\rho+k-1}} \sim \int_0^{\infty} \frac{d\phi}{(x+t)^{\rho+k-1}}, \quad x \rightarrow \infty$$

and by Theorem A it implies

$$(\rho)_k F \sim \phi, \quad x \rightarrow \infty.$$

Thus, we obtain that $F(x) \sim \frac{1}{(\rho)_k} x^{\rho+k-s-1} L(x)$, $x \rightarrow \infty$.

Since $\rho+k-s-1 > 0$, it follows ([1]) that F has the L -quasiasymptotic of the power $\rho+k-s-1$ and with the limit $\frac{1}{(\rho)_k} x^{\rho+k-s-1}$. Since $f = D^k F$ it easily follows

([1]) that f has the L -quasiasymptotic of the power $\rho-s-1$ and with the limit

$$\frac{(\rho+k-s-1)\dots(\rho-s)}{(\rho)_k} x^{\rho-s-1}. \tag{Δ}$$

COROLLARY. Let us suppose that $f \in J'(\rho)$ and for some $\tilde{\rho} > \rho$ $\tilde{\rho} \in \mathbb{R} \setminus (-N_0)$

$$(S_{\tilde{\rho}}\{f\})(x) \sim \frac{1}{x^s L(x)}, \quad x \rightarrow \infty, \quad s > 1,$$

where L is slowly varying function on $[A, \infty)$. Further, suppose $\tilde{\rho} - s + k > 0$ and $x^{\tilde{\rho} - s + k} L(x)$ is non-decreasing in $[A, \infty)$. (Consequently, $f \in I'(\tilde{\rho})$ and $f(x) = D^{k+1} \left(\int_0^x F(t) dt \right)$).

Then f has the L -quasiasymptotic of the power $\tilde{\rho} - 1 - s$ and with the limit

$$\frac{(\tilde{\rho} - s)}{k+1} \\ (\tilde{\rho}) \\ k+1$$

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